CHAPTER 1

VECTOR ALZEBRA & VECTOR CALCULUS

- **Vectors :** Physical quantities having both magnitude, a definite direction in space and it should follow the laws of vector addition.

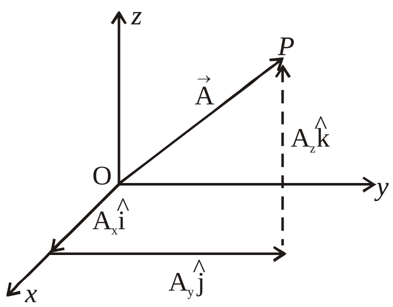
  Example : Velocity, Acceleration, Momentum, Force, Electric Field, Torque, etc.

- **Various type of vectors:**

1. **Equal vectors :** Vectors having same magnitude and same direction.
2. **Null Vectors :** Vector having coincident initial and terminal point i.e. its magnitude is zero.
3. **Unit Vectors :** Vectors having unit magnitude. Unit vector along $\vec{a} = \hat{a} = \frac{\vec{a}}{|\vec{a}|}$
4. **Reciprocal Vector:** Vector having same direction but reciprocal magnitude corresponding to original vector.
5. **Negative Vector :** Vectors having same magnitude but opposite direction corresponding to original vector.

- **Orthogonal Resolution of Vectors:**

  Any vector $\vec{A}$ in right- handed rectangular cartesian coordinate system can be represented as $\overrightarrow{OP} = \vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$, where, $\hat{i}, \hat{j}$ and $\hat{k}$ are the unit vectors in direction of x, y and z axis respectively and A_x, A_y, A_z are the rectangular components of vector $\vec{A}$ along x, y, z axis.



Magnitude of vector $\vec{A}$ is $|\vec{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Unit vector along $\vec{A}$ is $\hat{A} = \vec{A}/|\vec{A}| = \frac{A_x\hat{i} + A_y\hat{j} + A_z\hat{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$

If $\vec{A}$ makes angles α, β, γ with x, y, z axes respectively, then direction cosines of $\vec{A}$ are :

$l = \cos\alpha = \frac{A_x}{A}$; $m = \cos\beta = \frac{A_y}{A}$; $n = \cos\gamma = \frac{A_z}{A}$ and $l^2 + m^2 + n^2 = 1$

So, an unit vector along $\vec{A}$ can be written as $\hat{A} = l\hat{i} + m\hat{j} + n\hat{k}$

- **Scalar Product or Dot Product:** $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta \Rightarrow \cos\theta = \frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$

Properties:

(i) $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$, (ii) For two mutually perpendicular vectors $\vec{a}$ and $\vec{b}$, $\vec{a}\cdot\vec{b} = 0$,

(iii) $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$, $\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$

(iv) If $\vec{a} = a_z\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, then $\vec{a}\cdot\vec{b} = a_x b_x + a_y b_y + a_z b_z$

(v) Projection of $\vec{A}$ on $\vec{B} = \vec{A}.\hat{B}$

(vi) Work done by force $\vec{F}$ on an object in displacement of $\vec{r} = \vec{F}.\vec{r}$

- **Vector Product or Cross Product :** $\vec{a}\times\vec{b} = |\vec{a}||\vec{b}|\sin\theta\,\hat{n}$, where $\hat{n}$ is unit vector normal to the plane containing $\vec{a}$ and $\vec{b}$.

Properties:

(i) $\vec{a}\times\vec{b} = -(\vec{b}\times\vec{a})$ (ii) For two collinear vectors (parallel or anti-parallel vectors) $\vec{a}\times\vec{b} = 0$.

(iii) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$, $\hat{i}\times\hat{j} = \hat{k}$, $\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$

(iv) If $\vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$ and $\vec{b} = b_x\hat{i} + b_y\hat{j} + b_z\hat{k}$, then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

(v) Torque $(\vec{\tau}) = \vec{r}\times$(applied force)$\vec{F}$

(vi) Angular momentum $(\vec{L}) = \vec{r}\times$(linear momentum)$\vec{P}$

(vii) Linear velocity $(\vec{v}) =$ (angular velocity) $\vec{\omega}\times\vec{r}$

- **Scalar Triple Product:** $\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = [abc]$

Properties:

(i) $\vec{a}\cdot(\vec{b}\times\vec{c}) = \vec{b}\cdot(\vec{c}\times\vec{a}) = \vec{c}\cdot(\vec{a}\times\vec{b})$ i.e. [abc] = [bca] = [cab]

(ii) Volume of a parallelopiped having $\vec{a}, \vec{b}, \vec{c}$ as concurrent edges is :$V = \vec{a}\cdot(\vec{b}\times\vec{c})$

(iii) If $\vec{a}, \vec{b}, \vec{c}$ are coplanar vectors, then, [abc] = [bca] = [cab] = 0

- **Vector Triple Product:** $\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

It represents a vector coplanar with $\vec{b}$ and $\vec{c}$. and perpendicular to $\vec{a}$

SOLVED EXAMPLES

Example1: The range of x for which between the vectors $\vec{A} = 2x^2\hat{i} + 4x\hat{j} + \hat{k}$ and $\vec{B} = 7\hat{i} - 2\hat{j} + x\hat{k}$ is obtuse, is equal to

(a) x < 0 (b) x > 1/2 (c) 0 < x < 1/2 (d) None of these

Soln: Since, $90^0 < \theta < 180^0$, then $\cos\theta < 0 \Rightarrow \vec{A}.\vec{B} < 0 \Rightarrow 14x^2 - 7x < 0 \Rightarrow x(2x-1) < 0$

Therefore, $\boxed{0 < x < 1/2}$

Example 2: A unit vector $\hat{n}$ on the xy-plane is at an angle of 120º with respect to $\hat{i}$. The angle between the vectors $\vec{u} = a\hat{i} + b\hat{n}$ and $\vec{v} = a\hat{n} + b\hat{i}$ will be 60º if

(a) $b = \sqrt{3}a/2$ (b) $b = 2a/\sqrt{3}$ (c) b = a/2 (d) b = a

Soln: $\vec{u}.\vec{v} = |\vec{u}||\vec{v}|\cos 60^0$

$$\Rightarrow a^2(\hat{i}.\hat{n}) + b^2(\hat{n}.\hat{i}) + ab + ba = \sqrt{a^2 + b^2 + 2ab\cos 120^0}\sqrt{a^2 + b^2 + 2ab\cos 120^0}\,\frac{1}{2}$$

$$\Rightarrow a^2\cos 120^0 + b^2\cos 120^0 + ab + ba = (a^2 + b^2 - ab)\frac{1}{2}$$

$$\Rightarrow 2a^2 - 5ab + 2b^2 = 0 \Rightarrow a = \frac{b}{2} \text{ or } b = \frac{a}{2}$$

Example 3: The value of 'm' for which $\vec{A} = \hat{i} - \hat{j} - 2\hat{k}, \vec{B} = 3\hat{i} + 5\hat{j} + 6\hat{k}, \vec{C} = -\hat{i} + 4\hat{j} + m\hat{k}$ will be co-planar is

(a) 13/2 (b) 7/2 (c) 5/2 (d) 11/2

Soln: Since, $\vec{A}, \vec{B}, \vec{C}$ are co-planar, $\vec{A}.(\vec{B} \times \vec{C}) = 0 \Rightarrow \begin{vmatrix} 1 & -1 & -2 \\ 3 & 5 & 6 \\ -1 & 4 & m \end{vmatrix} = 0 \Rightarrow m = \frac{13}{2}$

Example 4: If $\vec{b} = \hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j}) + \hat{k} \times (\vec{a} \times \hat{k})$, then $\vec{b}$ can be simplified to

(a) 0 (b) $\vec{a}$ (c) $2\vec{a}$ (d) None of these

Soln: $\vec{b} = \hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j}) + \hat{k} \times (\vec{a} \times \hat{k})$

$$= \vec{a}(\hat{i}.\hat{i}) - \hat{i}(\hat{i}.\vec{a}) + \vec{a}(\hat{j}.\hat{j}) - \hat{j}(\hat{j}.\vec{a}) + \vec{a}(\hat{k}.\hat{k}) - \hat{k}(\hat{k}.\vec{a}) = 3\vec{a} - \vec{a} = 2\vec{a}$$

Example 5: Three unit vectors $\vec{a}, \vec{b}, \vec{c}$ ($\vec{b}$ and $\vec{c}$ are not parallel) are such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\sqrt{3}}{2}\vec{c}$. The angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$, respectively are

(a) $30^0, 90^0$ (b) $150^0, 90^0$ (c) $60^0, 90^0$ (d) $90^0, 30^0$

Soln: $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\sqrt{3}}{2}\vec{c} \Rightarrow \vec{b}(\vec{a}.\vec{c}) - \vec{c}(\vec{a}.\vec{b}) = \frac{\sqrt{3}}{2}\vec{c} \Rightarrow \vec{a}.\vec{c} = 0$ and $\vec{a}.\vec{b} = -\frac{\sqrt{3}}{2}$

The angle which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$ are $150^0, 90^0$ respectively.

- **Gradient of a Scalar Field :** Gradient of a continuously differentiable scalar function $\phi(x, y, z)$ is mathematically defined as:

$$\text{grad}\phi = \vec{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

where, $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k} \Rightarrow$ 'del' or 'nabla' operator

Physical interpretation : Gradient of scalar function ϕ at any pt. is a vector, whose magnitude is equal to the rate of change of scalar function ϕ with distance along the normal to level surface and its direction is along the normal to the level surface at that point.

Level surface : It has same value of scalar function at each point. Example: Equipotential surface.

Directional Derivative : Directional derivative of 'ϕ' in the direction of $\vec{A}$ is defined as the component of $\vec{\nabla}\phi$ in the direction of vector $\vec{A}$ and is given by, $\vec{\nabla}\phi \cdot \hat{A} = \vec{\nabla}\phi \cdot \frac{\vec{A}}{|\vec{A}|}$

Tangent Plane and Normal to the level surface:

Consider $\phi(x, y, z) = c$ be the equation of a level surface. and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of any point P (x,y,z) on this surface.

Tangent plane at P: $\vec{\nabla}\phi$ is a vector normal to the surface i.e. it is perpendicular to the tangent plane at P. Let, $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$ be the position vector of any point on the tangent plane at P to the surface. Therefore, $\vec{R} - \vec{r} = (X - x)\hat{i} + (Y - y)\hat{j} + (Z - z)\hat{k}$ lies in the tangent plane at P and it will be perpendicular to $\vec{\nabla}\phi$ i.e. $(\vec{R} - \vec{r}).\vec{\nabla}\phi = 0$

$$\Rightarrow (X - x)\frac{\partial\phi}{\partial x} + (Y - y)\frac{\partial\phi}{\partial y} + (Z - z)\frac{\partial\phi}{\partial z} = 0$$, which is the tangent plane at point P.

Normal at P: Let, $\vec{R} = X\hat{i} + Y\hat{j} + Z\hat{k}$ be the position vector of any point on the normal at P to the surface. Therefore, $\vec{R} - \vec{r} = (X - x)\hat{i} + (Y - y)\hat{j} + (Z - z)\hat{k}$ lies along the normal at P and it will be parellel to $\vec{\nabla}\phi$ i.e. $(\vec{R} - \vec{r}) \times \vec{\nabla}\phi = 0$, which is the vector equation of the normal at point P to the given surface.

- **Divergence of a vector field:** Divergence of a continuous differentiable vector point function $\vec{V}$ specified in a vector field is given by,

$$\vec{\nabla} \cdot \vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(V_x\hat{i} + V_y\hat{j} + V_z\hat{k}\right) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} = \text{Scalar quantity}$$

Physical Interpetation: Divergence of $\vec{A}$ at pt P(x, y, z) is defined as the inaward / outward flux of the vector field $\vec{A}$ per unit volume enclosed by an infinitesimal close surface surrounding point 'P'.

Properties:

(i) If $\vec{\nabla}.\vec{V}$ = 0, then $\vec{V}$ is known as “solenoidal Vector”.

(ii) If $\vec{\nabla}.\vec{V}$ = negative, then $\vec{V}$ is a sink field i.e. vector lines are going inward.

(iii) If $\vec{\nabla}.\vec{V}$ = positive, then $\vec{V}$ is a source field i.e. vector lines are the going outward.

(iv) $\vec{\nabla}\cdot(\vec{u}+\vec{v}) = \vec{\nabla}\cdot\vec{u}+\vec{\nabla}\cdot\vec{v}$

(v) $\vec{\nabla}\cdot(\vec{u}+\vec{v}) = (\vec{\nabla}u)\cdot\vec{u}+v(\vec{\nabla}\vec{u})$

- **Curl of a vector field:** Curl or rotation of a continuous differentiable vector point function $\vec{V}$ is given by,

$$\vec{\nabla}\times\vec{V} = \left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+k\frac{\partial}{\partial z}\right)\times\left(v_x\hat{i}+v_y\hat{j}+v_z\hat{k}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

It is the measurement of rotation of vector field.

Properties:

(i) If $\vec{\nabla}\times\vec{V}=0$, then $\vec{V}$ is an irrotational vector and we can write $\vec{V}=\vec{\nabla}\phi$

(ii) If $\vec{\nabla}\times\vec{V}\neq 0$, then $\vec{V}$ is a rotational vector.

(iii) $\vec{\nabla}\times(\vec{U}+\vec{V}) = \vec{\nabla}\times\vec{U}+\vec{\nabla}\times\vec{V}$

(iv) $\vec{\nabla}\times(u\vec{V}) = u\vec{\nabla}\times\vec{V}+(\vec{\nabla}u)\times\vec{V}$

- **Important Vector Identities:** If ϕ, ψ are scalar point functions and $\vec{A},\vec{B}$ are vector point functions in certain region then.

1. $\vec{\nabla}(\phi+\psi)=\vec{\nabla}\phi+\vec{\nabla}\psi$
2. $\vec{\nabla}(\phi\psi)=\phi\vec{\nabla}\psi+\psi\vec{\nabla}\phi$
3. $\vec{\nabla}\cdot(\vec{A}+\vec{B})=\vec{\nabla}\cdot\vec{A}+\vec{\nabla}\cdot\vec{B}$
4. $\vec{\nabla}\times(\vec{A}+\vec{B})=\vec{\nabla}\times\vec{A}+\vec{\nabla}\times\vec{B}$
5. $\vec{\nabla}(\vec{A}\cdot\vec{B})=\vec{A}\times(\vec{\nabla}\times\vec{B})+(\vec{A}\cdot\vec{\nabla})\vec{B}+\vec{B}\times(\vec{\nabla}\times\vec{A})+(\vec{B}\cdot\vec{\nabla})\vec{A}$
6. $\vec{\nabla}\times(\phi\vec{A})=\phi(\vec{\nabla}\cdot\vec{A})+\vec{A}\cdot\vec{\nabla}\phi$
7. $\vec{\nabla}\times(\phi\vec{A})=\phi(\vec{\nabla}\times\vec{A})+(\vec{\nabla}\phi)\times\vec{A}$
8. $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A})=0$
9. $\vec{\nabla}\times(\vec{\nabla}\phi)=0$
10. $\vec{\nabla}\times(\vec{A}\times\vec{B})=\vec{B}\cdot(\vec{\nabla}\times\vec{A})-\vec{A}\cdot(\vec{\nabla}\times\vec{B})$
11. $\vec{\nabla}\times(\vec{A}\times\vec{B})=(\vec{B}\cdot\vec{\nabla})\vec{A}-(\vec{A}\cdot\vec{\nabla})\vec{B}+\vec{A}(\vec{\nabla}\cdot\vec{B})-\vec{B}(\vec{\nabla}\cdot\vec{A})$
12. $\vec{\nabla}\times(\vec{\nabla}\times\vec{A})=\vec{\nabla}(\vec{\nabla}\cdot\vec{A})-\nabla^2\vec{A}$