Chapter–1

Functions

1.1 FUNCTIONS - I

INTRODUCTION TO FUNCTIONS

Functions provide us with a convenient way to handle the relationship between the values of one variable quantity that depends on the values quantity. For example, let us imagine a spherical rubber balloon into which air is being pumped. The radius of the balloon (r) is changing with time t. In mathematics, we say that r is a function of time t and symbolically, it may be written,

$r = f(t) : r$ is a function of time t.

Similarly the volume of the balloon also depends on time t. Hence we can write

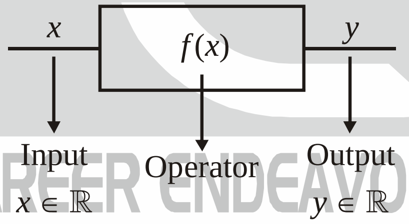
$V = g(t) : V$ is function of time t.

Different letters f and g are used because they represent different mathematical relations.
In general, if the values of a variable y depend on the values of another variable x, we write

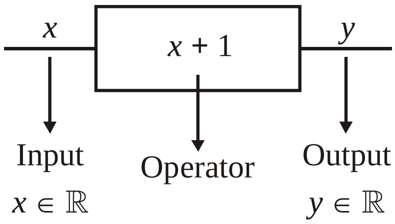
$y = f(x)$ *i.e.*, y is a function of x.

Explanation : Let $y = f(x)$ be a real valued function.

Here $f(x)$ can be understood as an operator between x and y. 'x' is feeded as input and 'y' is the corresponding output.



Consider $y = f(x) = x + 1$



i.e.,

x (input)	0	–3	1
y (output)	1	–2	2

Ordered Pair : The combination of input and output is called an ordered pair.

Representation of Ordered Pair : $(x, \; y)$

$\downarrow \quad \downarrow$

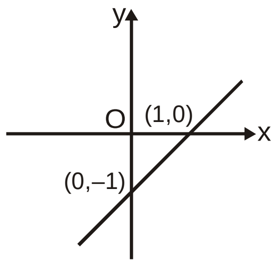
input output

So in above example ordered pairs are (0, 1), (–3, –2), (1, 2) etc., which satisfy the function.

GRAPH OF A FUNCTION

It is the pictorial representation of a function. It is formed by plotting ordered pairs that satisfy function.

Let $y = f(x) = x - 1$. Graph of $y = f(x)$ is shown below:



$y = f(x)$ is a linear polynomial whose graph is a straight line.

Note: A unique line passes through two given points. So to draw the graph of linear polynomials we needed to plot only two ordered pairs and join them.

Illustrating the Concept :

1. Suppose it is given that *y* is the square of *x*. Then we may write, $y = f(x) = x^2$

2. Velocity of uniformly accelerating particle starting from rest depends on time *t*.
If acceleration is 2 m/s^2, we can write, velocity (v) as a function of *t*.
i.e., $v(t) = 0 + 2t$ {Using $v = u + at$ }

INTERVALS AND NOTATIONS

To express values a variable can take, we use the following notations.

(i) Open interval :
If *x* can take values which lie strictly between *a* and *b* then we can write, $a < x < b$ or $x \in (a, b)$

(ii) Closed interval :
If *x* can take values which lie strictly between *a* and *b* or *x* can be equal to *a* or *x* can be equal to *b*, then we can write, $a \le x \le b$ or $x \in [a, b]$

(iii) Half-open interval :
If only one end point is included for values of *x*, then the interval is called as half-open interval.

$a < x \le b$ or $x \in (a, b]$

$a \le x < b$ or $x \in [a, b)$

(iv) Infinite intervals :
If x can take all real values, than we can write as

$-\infty < x < \infty \quad \Rightarrow \quad x \in (-\infty, \infty)$ or $x \in R$

(v) Other Notations :

$a < x \quad \Rightarrow \quad x \in (a, \infty)$

$a \le x \quad \Rightarrow \quad x \in [a, \infty)$

$x < b \quad \Rightarrow \quad x \in (-\infty, b)$

$x \le b \quad \Rightarrow \quad x \in (-\infty, b]$

(vi) If x can take specific values, say $x = a,\ x = b$ and $x = c$, then we can write, $x \in \{a, b, c\}$

1.1.1 DOMAIN

While defining real-valued functions, we have to observe some restrictions. One such restriction is that we can never divide by zero (0). Hence in the function

$y = f(x) = \dfrac{1}{x-1}$, x cannot be equal to 1.

Domain of $y = f(x)$ is collection of all inputs that operator can take so that output of operator exists

OR

The set of values of x for which y takes real values (so that the function is well defined) is known as the set of Domain for that function.

Hence the domain of $y = f(x)$ in above example is $x \in (-\infty, 1) \cup (1, \infty)$ or $x \in R - \{1\}$.

Example-1

Find the domain of the following functions.

(a) $y = \sqrt{1 - x^2}$ (b) $y = 2\sin x$ (c) $y = \dfrac{1}{x-2}$

Soln. (a) Square root of a negative number is not defined

$\therefore \quad 1 - x^2 \geq 0$

$\Rightarrow \quad x^2 \leq 1$

$\Rightarrow \quad -1 \leq x \leq 1$

Hence the domain of x is the interval $[-1, 1]$.

(b) $y = 2\sin x$

Trigonometric function $\sin x$ is defined for all values of x

$\therefore \quad x \in R$ i.e. $x \in (-\infty, \infty)$

(c) As denominator cannot be zero, x can not be 2

$\therefore \quad x \in (-\infty, 2) \cup (2, \infty)$ i.e. $x \in R - \{2\}$

Example-2

Find the domain of the following function.

(a) $\sqrt{3-x} + \dfrac{1}{\log_{10} x}$ (b) $\dfrac{1}{x + |x|}$ (c) $\sqrt{1 - \log_{10} x}$

Soln. (a) Square root of a negative number is not defined

$\therefore \quad \sqrt{3-x}$ is defined if $3 - x \geq 0$

$\Rightarrow \quad x \leq 3 \qquad$ .... (i)

$\log_{10} x$ is defined if $x > 0$ and also denominator cannot be 0

$\therefore \quad \log_{10} x$ cannot be 0 $\Rightarrow \quad x \neq 1$

Hence $\dfrac{1}{\log_{10} x}$ is defined if $x > 0$ and $x \neq 1$

$\Rightarrow \quad x \in (0, \infty) - \{1\} \qquad$ .... (ii)

Combining (i) and (ii), set of domain is $x \in (0, 1) \cup (1, 3]$.

(b) $f(x)$ is defined if : $x + |x| \neq 0$

$\Rightarrow \quad |x| \neq -x \Rightarrow x > 0 \quad \left\{ \text{Using } |x| = \begin{cases} x & \text{if } x > 0 \\ -x & \text{if } x \leq 0 \end{cases} \right\}$

Hence domain is $x \in (0, \infty)$.

(c) $f(x)$ is defined if $1 - \log_{10} x \geq 0$ and $x > 0$

$\Rightarrow \quad \log_{10} x \leq 1$ and $x > 0$

$\Rightarrow \quad x \leq 10$ and $x > 0$

$\Rightarrow$ domain is $x \in (0, 10]$

RANGE

Range of $y = f(x)$ is collection of all outputs $\{f(x)\}$ corresponding to each real number in the domain.

OR

The set of values which y can take is known as the set of Range for that function.

1.1.2 NATURE OF A FUNCTION

(*i*) EVEN FUNCTION

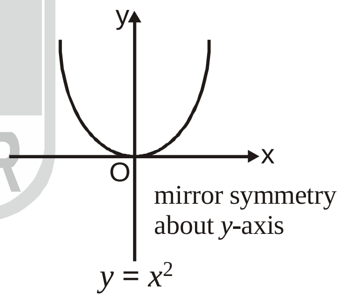
If a function $y = f(x)$ satisfies $f(-x) = f(x)$ for all values of x, then $y = f(x)$ is called an even function.

Note: As an even function satisfies, $f(-x) = f(x)$, $f(x)$ possesses same value for value of x which are equal in magnitude and opposite in sign.

For example : $f(-1) = f(1)$, $f(-2) = f(2)$, $f(-3) = f(3)$......

Therefore graph of an even function is symmetrical about y-axis i.e. left half is mirror image of right half and right half is mirror image of left half, considering y-axis as mirror.

Illustrating the Concept :



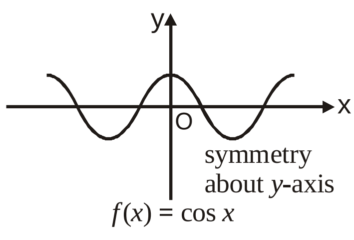
(i) Consider $f(x) = x^2$

$f(-x) = (-x)^2 = x^2$

$\Rightarrow \quad f(-x) = f(x)$

Hence $f(x) = x^2$ is an even function.

Note: Graph of $f(x) = x^2$ is symmetrical about y-axis.



(ii) Consider $f(x) = \cos x$

$f(-x) = \cos(-x) = \cos x$ [using $\cos(-\theta) = \cos\theta$]

$\Rightarrow \quad f(-x) = f(x)$

Hence $f(x) = \cos x$ is an even funtion.

Note: Graph of $f(x) = \cos x$ is symmetrical about y-axis.

(*ii*) ODD FUNCTION

If a function $y = f(x)$ satisfies, $f(-x) = -f(x)$ for all values of x, then $y = f(x)$ is called an odd function.

Note: As an odd function satisfies satisfies, $f(-x) = -f(x)$, $f(x)$ possesses values equal in magnitude but opposite in sign for all values of x which are also equal in magnitude but with opposite signs.

For example : $f(-1) = -f(1)$, $f(-2) = -f(2)$, $f(-3) = -f(3)$......

Therefore graph of an odd function is symmetrical about origin i.e. if we rotate the graph in right half about origin through 180°, then we get graph in left half.

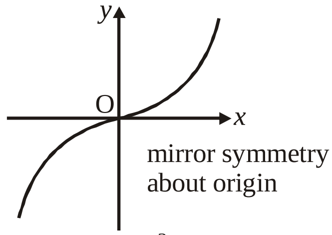
Illustrating the Concept :

(i) Consider $f(x) = x^3$

$$f(-x) = (-x)^3 = -x^3$$

$\Rightarrow$ $f(x)$ satisfies, $f(-x) = -f(x)$

Hence $f(x) = x^3$ is an odd function.



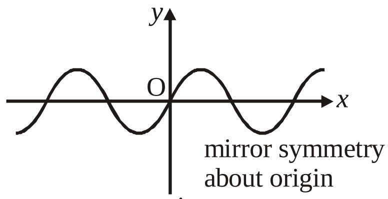
$f(x) = x^3$

(ii) Consider $f(x) = \sin x$

$$f(-x) = \sin(-x) = -\sin x \quad [\text{using } \sin(-\theta) = -\sin\theta]$$

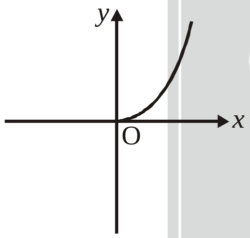
$\Rightarrow$ $f(x)$ satisfies, $f(-x) = -f(x)$

Hence $f(x) = \sin x$ is an odd function.

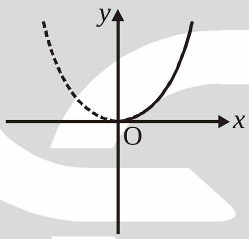


$y = \sin x$

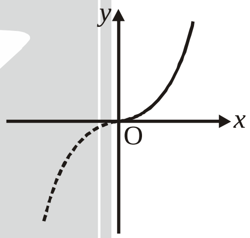
Note: As graph is symmetrical about origin, left half of the graph can also be drawn by taking reflection of right half in both x-axis as well as y-axis.



Right half of
$y = x^3$



Reflection of right
half in y-axis



Reflection of the reflected
part in x-axis

PROPERTIES OF EVEN AND ODD FUNCTION

(a) Sum :
- **(i)** even + even = even
- **(ii)** even + odd = neither even nor odd
- **(iii)** odd + even = neither even nor odd
- **(iv)** odd + odd = odd

(b) Difference :
- **(i)** even – even = even
- **(ii)** even – odd = neither even nor odd
- **(iii)** odd – even = neither even nor odd
- **(iv)** odd – odd = odd

(c) Product :
- **(i)** even × even = even
- **(ii)** even × odd = odd
- **(iii)** odd × even = odd
- **(iv)** odd × odd = even

(d) Division :
- **(i)** even ÷ even = even
- **(ii)** odd ÷ even = odd
- **(iii)** odd ÷ even = odd
- **(iv)** odd ÷ odd = even

(e)
- **(i)** if $f(x) + f(-x) = 0 \Rightarrow f$ is odd function
- **(ii)** if $f(x) - f(-x) = 0 \Rightarrow f$ is even function

(f) The derivative of an odd function is an even function and derivative of an even function is an odd function.

(g) The square of even or an odd function is always an even function.

(h) Any function $y = f(x)$ can written as $y = f(x) = [\text{ odd part of } f(x)] + [\text{even part of } f(x)]$

i.e., $y = f(x) = \left[\dfrac{f(x) - f(-x)}{2}\right] + \left[\dfrac{f(x) + f(-x)}{2}\right]$

(*iii*) PERIODIC FUNCTION

A function $f(x)$ is said to be periodic function of x, if there exists a positive real number T such that $f(x+T) = f(x)$. The smallest value of T is called the period of the function.

Note: The positive T should be independent of x for $f(x)$ to be periodic. In case T is not independent of x, $f(x)$ is not a periodic function.

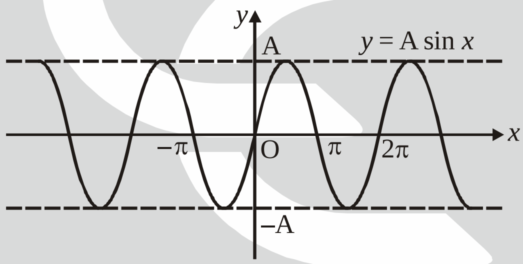
DEFINITION (GRAPHICALLY)

A function is said to be periodic if its graph repeats itself after a fixed interval and the width of that interval is called its period.

For example :

$\because$ Graph of $f(x) = A\sin x$ repeats after an interval of 2π.

Thus, $f(x) = A\sin x$ is periodic with period 2π.



Standard Result on Periodic Functions

1.	$\sin^n x, \cos^n x$ $\sec^n x, \text{cosec}^n x$	π, If n is even 2π, If n is odd or fraction
2.	$\tan^n x, \cot^n x$	π, n is even or odd
3.	$\lvert\sin x\rvert, \lvert\cos x\rvert$ $\lvert\tan x\rvert, \lvert\cot x\rvert$ $\lvert\sec x\rvert, \lvert\text{cosec } x\rvert$	π π π
4.	$x - [x] = \{x\}$	1
5.	$\sqrt{x}, x^2, x^3 + 5$ etc.	Period does not exist

PROPERTIES OF A PERIOD FUNCTION

(i) If $f(x)$ has period T, then

(a) $cf(x)$ is periodic with period T

(b) $f(x+c)$ is periodic with period T

(c) $f(x) \pm c$ is periodic with period T

(d) If constant is added, subtracted, multiplied or divided in periodic function, period remains same

(e) Every constant function is always periodic, with no fundamental period

(f) Inverse of a periodic function does not exist. But in case of trigonometric function since domain and range are restricted and defined, hence inverse exists.

(ii) If $f(x)$ is periodic with period T, then $kf(cx+d)$ has period $T/|c|$, hence period is affected by coefficient of x only.

(iii) If $f(x)$ and $g(x)$ are two functions with period T_1 and T_2 respectively and $h(x) = af(x) + bg(x)$ then $h(x)$ has period = LCM of $\{T_1, T_2\}$

Note: There are some exceptions to above result :

For example :

Period of $\{f(x) = |\sin x| + |\cos x|\} = \pi/2$ instead of π,

Period of $\{f(x) = \sin^4 x + \cos^4 x\} = \pi/2$ instead of π,

Period of $\{f(x) = |\tan x| + |\cos x|\} = \pi/2$ instead of π

(v) If $f(x)$ is a periodic function with period T and $g(x)$ is any function such that range of f is a proper subset of domain of g, then $g(f(x))$ is periodic with period T.

For example : $\sin(x-[x]) = \sin(\{x\})$ is periodic with period 1 as $x-[x]$ is periodic with period 1.

HOW TO MAKE LCM

1. LCM of $\left(\frac{p}{q}, \frac{r}{s}, \frac{t}{u}\right) = \frac{\text{LCM of } (p, r, t)}{\text{HCF of } (q, s, u)}$

2. (a) LCM of rational with rational is possible.

(b) LCM of irrational with irrational is possible but LCM of two irrational number of different kind (for example $2\sqrt{3}$ and $3\sqrt{2}$) does not exist.

(c) LCM of rational with irrational is not possible.

For example : LCM of $(2\pi, 2, 6\pi)$ is not possible as $2\pi, 6\pi \in$ Irrational and $2 \in$ rational.

1.1.3 CLASSIFICATION OF FUNCTIONS

BASIC FUNCTIONS CAN BE CATEGORISED INTO THE FOLLOWING CATEGORIES.

1. Algebraic functions

Algebraic functions can be of the following types :

(a) Monomial function (b) Polynomial function
(c) Rational function (d) Irrational function

2. Transcendental functions

Transcendental functions can be of the following types :

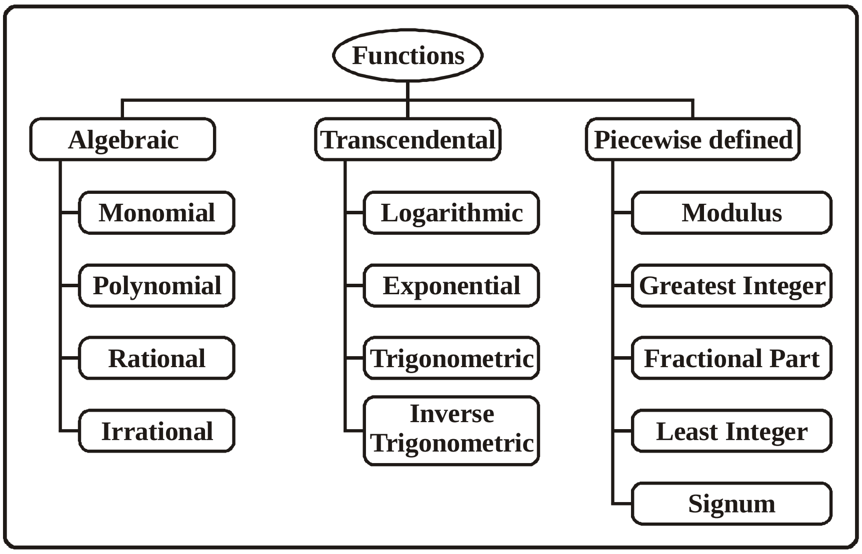
(a) Logarithmic function (b) Exponential function
(c) Trigonometric function (d) Inverse Trigonometric function

3. Piecewise defined functions

Piecewise defined functions can be of the following types :

(a) Modulus function (b) Greatest Integer function
(c) Fractional Part function (d) Least Integer function
(e) Signum function

Following chart shows relationship between these functions.



(*i*) ALGEBRAIC FUNCTIONS

(a) Monomial function

Any function of the form $f(x)=kx^n$, where k is constant and $n \in N$ is known as monomial function.

For example :

$f(x)=3x^4$ is a monomial function of degree 4.

$f(x)=-2x^3$ is a monomial function of degree 3.

(b) Polynomial function

A function $f(x)=a_0x^n+a_1x^{n-1}+\cdots+a_n, \quad (a_0 \neq 0)$

where $a_0, a_1, a_2, ..., a_n$ are real constant and n is non-negative integer, then $f(x)$ is called polynomial function, where n is the degree of polynomial function.

For example :

$f(x)=x^{1920}+5x^{1919}+6x$ (polynomial of degree 1920)

$g(x)=x^2+3x+3$ (polynomial of degree 2)

$h(x)=7=7x^0$ (polynomial of degree 0)

(i) Constant function : If degree of a polynomial function is 0, then polynomial function is called as constant function.

(ii) Identity function : If polynomial function takes the form $y=f(x)=x$ for all $x \in R$, then it is called as identity function on R.

(c) Rational algebraic function : A function of the form $f(x)=\dfrac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials and $q(x)\neq 0$, is called a rational function.

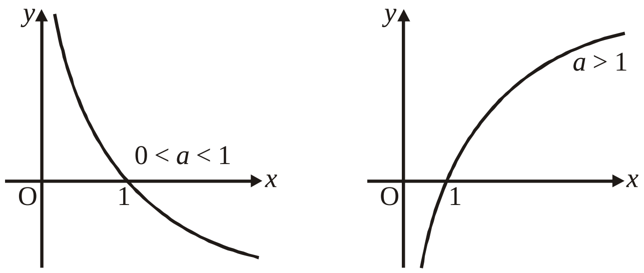
(d) Irrational function : An algebraic function or rational function containing one or more radicals (non-integral rational powers of x) is called an irrational function.

e.g., $\dfrac{x^3-\sqrt{x}}{2x^2-9}, 3x^3-x^{3/2}+9x-1$

(*ii*) TRANSCENDENTAL FUNCTION

(a) Logarithmic function

Logarithmic function is represented as $y=\log_a x$ where $x>0,\ a\in(0,1)\cup(1,\infty)$. If $a>1$, y increases as x increases (as seen from graph). If $0 < a < 1$, y decreases as x increases.



Continuity : The graph of $f(x) = \log_a x$ is continuous (i.e. no break in the curve) in the respective domain.

Domain and Range : The Domain of the function $f(x) = \log_a(x)$ is $x > 0$ and Range is $y \in R$.

FUNCTIONS

(a) Properties of Logarithmic Function

(i) $\log_a a = 1$

(ii) $\log_a 1 = 0$

(iii) $\log_a(mn) = \log_a m + \log_a n$

(iv) $\log_a(m/n) = \log_a m - \log_a n$

(v) $\log_a x^m = m \log_a x$

(vi) $\log_a b = \dfrac{\log_m b}{\log_m a}$ $m \neq 1, m > 0$

(vii) $a^{\log_a x} = x$

(viii) $a^{\log_c b} = b^{\log_c a}$

(xi) If $\log_m x > \log_m y \Rightarrow \begin{cases} x > y, & \text{if} \quad m > 1 \\ x < y, & \text{if} \quad 0 < m < 1 \end{cases}$

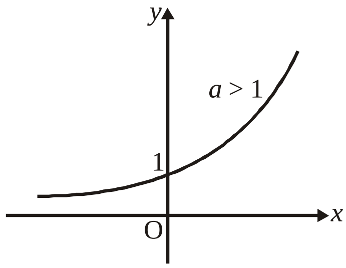
(x) $\log_m a = b \Rightarrow a = m^b$

(xi) $\log_m a > b \Rightarrow \begin{cases} a > m^b, & \text{if} \quad m > 1 \\ a < m^b, & \text{if} \quad 0 < m < 1 \end{cases}$

(xii) $\log_m a < b \Rightarrow \begin{cases} a < m^b, & \text{if} \quad m > 1 \\ a > m^b, & \text{if} \quad 0 < m < 1 \end{cases}$

(b) Exponential function : $y = a^x$ where $a > 1$ or $0 < a < 1$ is an exponential function of x.

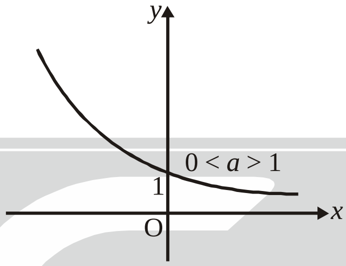
This function is the inverse of logarithmic function i.e. it can be obtained by interchanging x and y in $y = \log_a x$.



As observed from the graph, if $a > 1$, then y increases as x increases. If $0 < a < 1$, then y decreases as x increases.

Continuity : The graph of $f(x) = a^x$ is continuous (i.e. no break in the curve) everywhere.

Domain and Range : The domain of the function $f(x)$ is $x \in R$ and Range is $y > 0$.

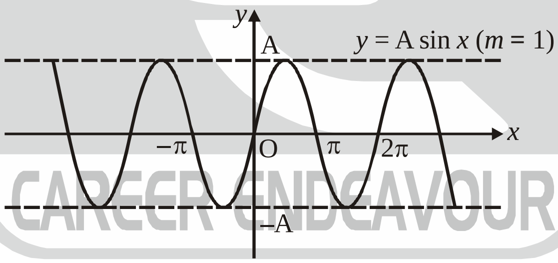


(c) Trigonometric function

(i) $y = A \sin (mx)$

(ii) $y = A \cos (mx)$

(iii) $y = A \tan (mx)$

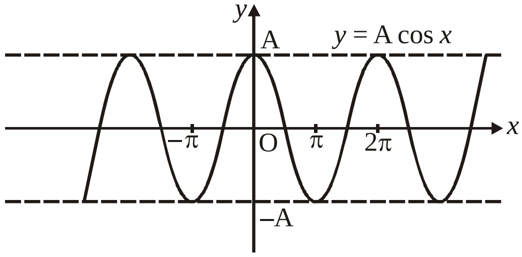


Period :

Period of $y = A \sin (mx)$ and $y = A \cos (mx)$ is $T = (2\pi)/m$

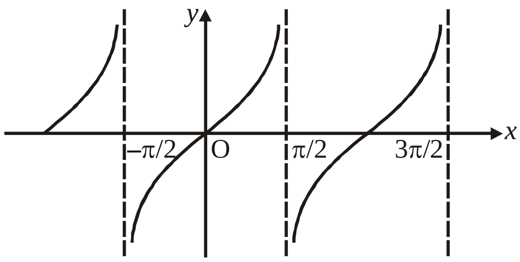
Period of $y = A \tan (mx)$ is $T = \pi/m$

Continuity : The graph of $y = A \sin (mx)$ and $y = A \cos (mx)$ is continuous (i.e., no break in the curve) every where. The graph of $y = A \tan (mx)$ is discontinuous (i.e. break in the curve) at $x = (2n+1)\dfrac{\pi}{2m}$.



Domain and Range : The domain of $y = A \sin (mx)$ and $y = A \cos (mx)$ is $x \in R$ and Range is set [–

A, A]. The Domain of $y = A\tan(mx)$ is $x \in R - (2n+1)\dfrac{\pi}{2m}$ and Range is $y \in R$.

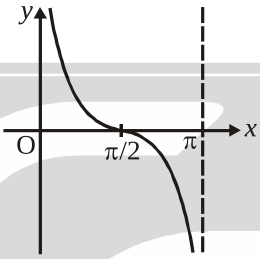


y = A tan x

(iv) $y = A \cot mx$

Period : It is a periodic function with period $= \pi/m$.

Continuity : It can be observed that $y = A\cot mx$ is discontinuous at $x = n\pi/m$ where $n \in I$



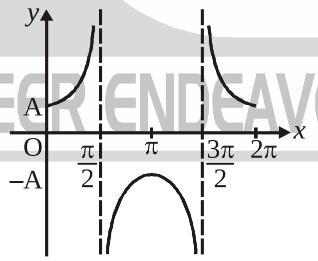
y = A cot x

Domain and Range : The domain of $y = A\cot mx$ is $x \in R - \dfrac{n\pi}{m}$ and the range is $y \in R$.

(v) $y = A \sec mx$

Period : It is a periodic function with period $= 2\pi/m$.

Continuity : It can be observed that $y = A\sec mx$ is discontinuous at $x = (2n+1)\pi/2m, n \in I$.



y = A sec x

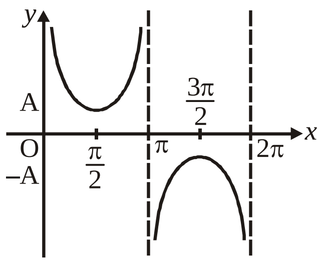
Domain and Range : The domain of $y = A\sec mx$ is $x \in R - (2n+1)\dfrac{\pi}{2m}$ and the Range is

$y \in (-\infty, -A] \cup [A, \infty)$.

(vi) $y = A \operatorname{cosec} mx$

Period : It is a periodic function with period $2\pi/m$.

Continuity : It can be observed that $y = A \operatorname{cosec} mx$ is discontinuous at $x = n\pi/m$.



y = A cosec x

Domain and Range : The domain of $y = A \operatorname{cosec} mx$ is $R - \{n\pi/m\}$ and the range is $y \in (-\infty, -A] \cup [A, \infty)$.

(d) Inverse Trigonometric Function

(To be discussed in the later module of function).

(*iii*) PIECEWISE DEFINED FUNCTIONS

(a) Modulus or Absolute Value Function

Modulus function is a numerical value function or we also can it as absolute value function.

$|5| = |-5| = 5, |-1| = 1, |-2.53| = 2.53$ ... etc.

It can also be understood as the distance defined with respect to origin.

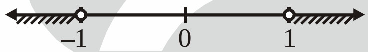
For example, if $|x| = 1$ means **distance covered is one unit** on right hand side or left side of origin.

$\therefore \quad |x| = 1$

$\Rightarrow \quad x = \pm 1$

if, $|x| < 1$ means **distance covered is less than one unit** on right hand side or left side of origin as shown in the following figure.

Similarly $|x| > 1$, means **distance covered is more than one unit** on right hands side or left of origin as shown in the following figure.

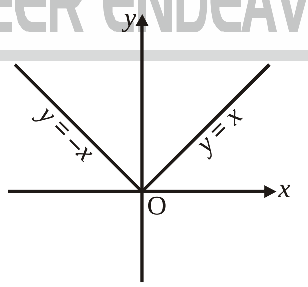


Modulus of *x*

$f(x) = |x| =$ magnitude of *x* or the positive value of *x*.

The expression $|x|$ can be further split as follows :

$$y = |x| = \begin{cases} x & ; \quad x \ge 0 \\ -x & ; \quad x < 0 \end{cases}$$



Continuity : The graph of $y = |x|$ is continuous (i.e. no break in the curve) but has a corner at origin as shown.

Domain and Range : The domain of the function $f(x)$ is $x \in R$ and Range is $y \in [0, \infty)$.

Results :

(A) (i) $|f(x)| = a \quad \Rightarrow \quad f(x) = \pm a$

(ii) $|f(x)| < a \quad \Rightarrow \quad -a < f(x) < a$

(iii) $|f(x)| > a \quad \Rightarrow \quad f(x) < -a$ or $f(x) > a$

(B) If $a < 0$, then :

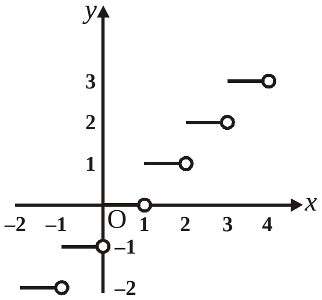
(i) $|f(x)| = a \Rightarrow$ no solution

(ii) $|f(x)| < a \Rightarrow$ no solution

(iii) $|f(x)| > a \Rightarrow$ all real values of x in domain of $f(x)$

(b) Greatest integer function (unit step function)

$y = [x] =$ the greatest integer less than or equal to x.



It can also be simplified as :

$y = [x] = n$ if $n \le x < n+1$, where n is an integer.

Continuity : The graph of $f(x)$ is discontinuous (i.e. break in the curve) at integral values of x.

Domain and Range : The Domain of the function $f(x)$ is $x \in R$ and Range is $y \in I$ (integer).

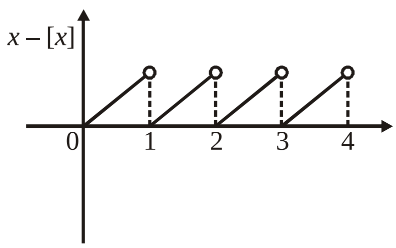
Note:
(a) $[x] = x$ holds if x is integer
(b) $[x+a] = [x] + a$ if a is integer
(c) $x = [x] + \{x\}$, $\{x\}$ denotes the fractional part of x
(d) $[-x] = -[x],\ x \in I$
(e) $[-x] = -[x] - 1,\ x \notin I$
(f) $[x+y] \ge [x] + [y]$
(g) $\left[\frac{[x]}{n}\right] = \left[\frac{x}{n}\right], n \in N$

(c) Fractional part function

Fractional part function is represent as $y = \{x\} = x - [x]$

If we express $y = \{x\}$ in intervals $\forall\ x \in [0, 4]$, we get:

$$y = \{x\} = x - [x] = \begin{cases} x & 0 \le x < 1 \\ x-1 & 1 \le x < 2 \\ x-2 & 2 \le x < 3 \\ x-3 & 3 \le x < 4 \\ 0 & x = 4 \end{cases}$$



Now plot the graph of above definition as shown in the figure.
We can extend graph for other values of x.

Continuity : If we observe graph, we can see that graph has breaks at all integer values. Hence $y = \{x\}$ is discontinuous $\forall\ x \in I$.

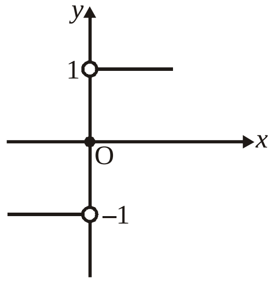
Periodicity : From graph, we can see that $y = \{x\}$ repeats after interval 1. Therefore $y = \{x\}$ is a periodic function with period 1.

Domain and Range : Domain of $y=\{x\}$ is $x \in R$ and range is $y \in [0,1)$.

(d) Least integer function : (To be discussed in the later module of functions).

(e) Signum function

$$y=\operatorname{sgn}(x)=\begin{cases} 1 & ; \quad x>0 \\ 0 & ; \quad x=0 \\ -1 & ; \quad x<0 \end{cases}$$



This can also be written as

$$y=\operatorname{sgn}(x)=\begin{cases} |x|/x & ; \quad x \neq 0 \\ 0 & ; \quad x=0 \end{cases}$$

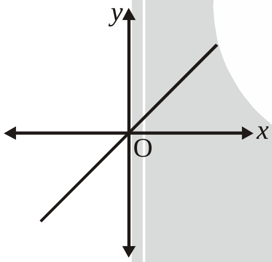
Domain and Range : Domain of $y=\operatorname{sgn}(x)$ is $x \in R$ and range is $y \in \{-1, 0, 1\}$

Continuity : The graph of $f(x)$ is continuous for all values of x except at $x=0$ where there is a break in the curve.

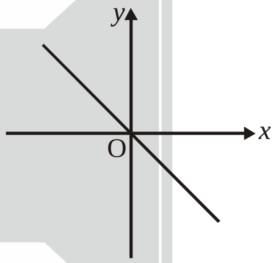
1.1.4 GRAPHS TO REMEMBER

You are supposed to remember graphs of all functions that we studied in earlier sections. There are other graphs that you should learn and remember. See the following graphs that you have to always remember.

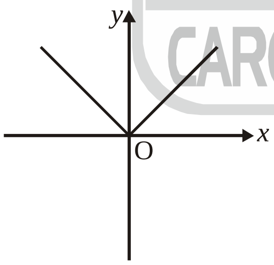
1. $y=x$



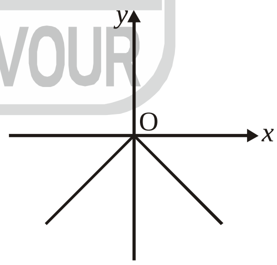
2. $y=-x$



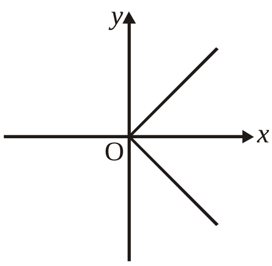
3. $y=|x|$



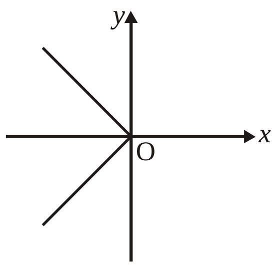
4. $y=-|x|$



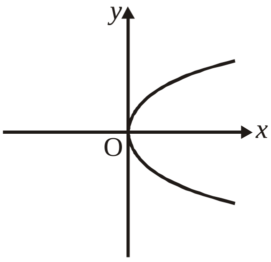
5. $|y|=x$



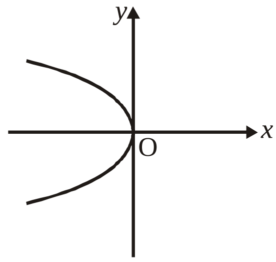
6. $-|y|=x$



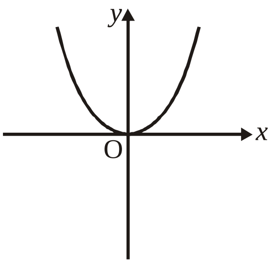
7. $y^2 = 4ax$



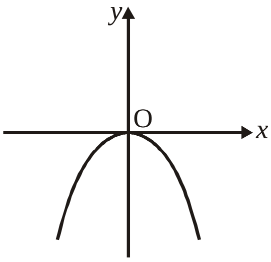
8. $y^2 = -4ax$



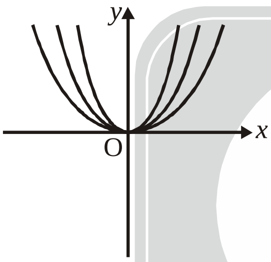
9. $x^2 = 4ay$



10. $x^2 = -4ay$

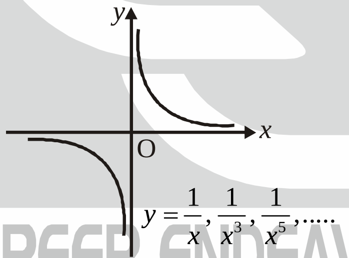


11. $y = ax^2$



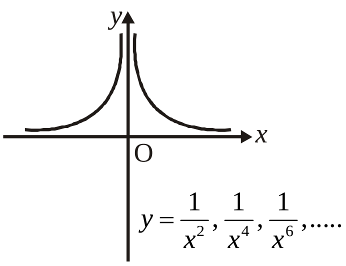
greater is 'a' steeper is curve.

12. $y = \frac{1}{x^n}$, n is odd integer ≥ 1



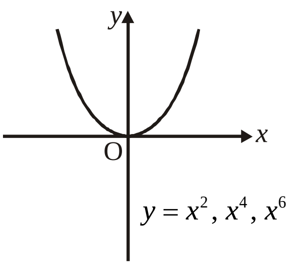
[i.e. graph of $y = \frac{1}{x}, \frac{1}{x^3}, \frac{1}{x^5}, ...$ can be considered same (for rough sketching)].

13. $y = \frac{1}{x^n}$, n is even integer > 1



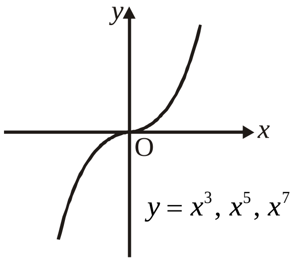
[i.e. graph of $y = \frac{1}{x^2}, \frac{1}{x^4}, \frac{1}{x^6}, ...$ can be considered same (for rough sketching)].

14. $y = x^n$, n is even integer ≥ 2



[i.e. graph of $y = x^2, x^4, x^6, ...$ can be considered same (for rough sketching)].

15. $y = x^n$, n is odd integer > 1



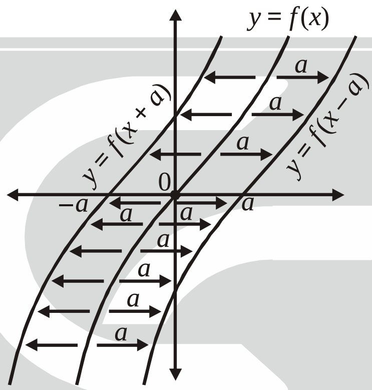
[i.e. graph of $y = x^3, x^5, x^7, ...$ can be considered same (for rough sketching)].

1.1.5 TRANSFORMATION

(i) Transformation 1

(a) $y = f(x) \longrightarrow y = f(x+a)$

To draw $y = f(x+a)$, shift the graph of $y = f(x)$ through 'a' units towards left.



(b) $y = f(x) \longrightarrow y = f(x-a)$

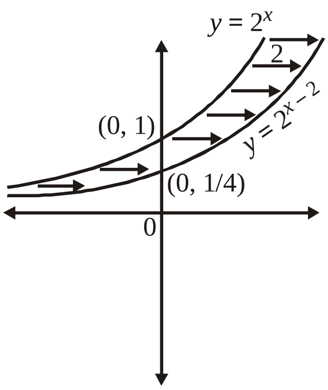
To draw $y = f(x-a)$ shift the graph of $y = f(x)$ through 'a' unit towards right.

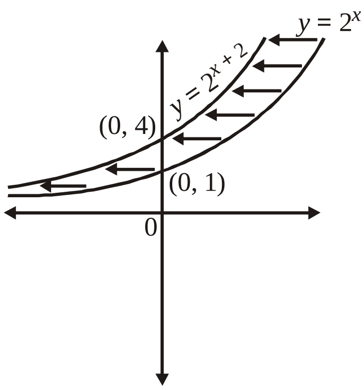
Example-1

Plot the graph of following :

(a) $4y = 2^x$ **(b)** $y = 4.2^x$

Soln. **(a)** $4y = 2^x \Rightarrow y = 2^{x-2}$ **(b)** $y = 4.2^x \Rightarrow y = 2^{x+2}$

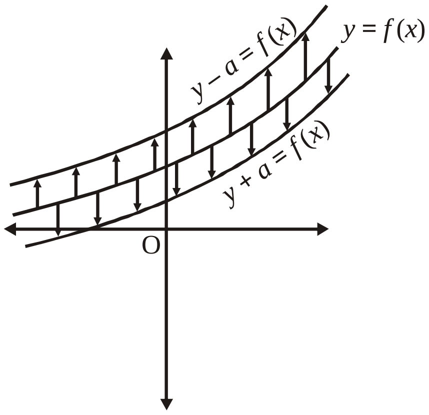




(ii) Transformation 2

(a) $y = f(x) \longrightarrow y + a = f(x)$

To draw $y + a = f(x)$, shift graph of $y = f(x)$ by 'a' units downward.



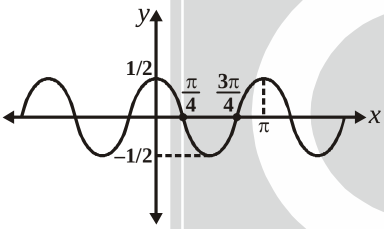
(b) $y = f(x) \longrightarrow y - a = f(x)$

To draw $y - a = f(x)$, shift the graph of $y = f(x)$ by 'a' units upward.

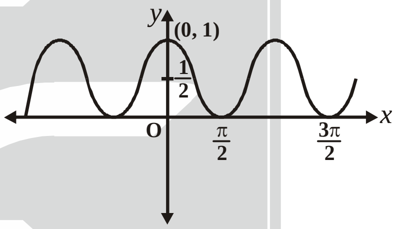
Example-2

Draw the graph of : $y = \cos^2 x$.

Soln. $y = \dfrac{1+\cos 2x}{2} \Rightarrow y = \dfrac{1}{2} + \dfrac{\cos 2x}{2} \xleftarrow[y-a=f(x)]{} y = \dfrac{\cos 2x}{2}$



Graph of $y = \dfrac{\cos 2x}{2}$



Graph of $y - \dfrac{1}{2} = \dfrac{\cos 2x}{2} \Rightarrow y = \cos^2 x$

Shift 1/2 unit up

Observation:

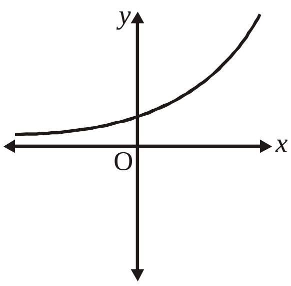
In $f(x) = \cos^2 x$, maximum value of $f(x)$ is 1 and minimum value of $f(x)$ is 0.

Example-3

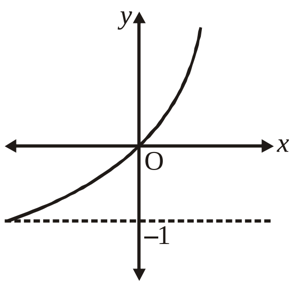
Plot the graph of following :

(a) $y = e^x - 1$ **(b)** $y = \log_e x - 1$

Soln. **(a)** $y = e^x - 1 \Rightarrow y + 1 = e^x \xleftarrow[y+a=f(x)]{} y = e^x$



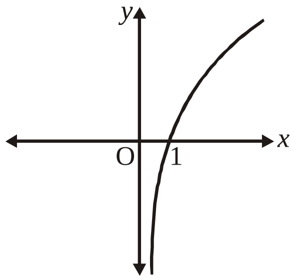
Graph of $y = e^x$



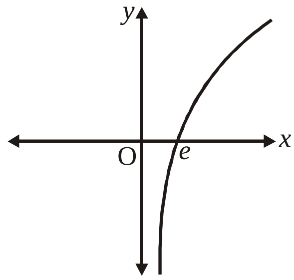
Graph of $y + 1 = e^x$

Shift 1 unit down

(b) $y = \log_e x - 1 \Rightarrow y + 1 = \log_e x \xleftarrow[y+a=f(x)]{} y = \log_e x$



Graph of $y = \log_e x$



Graph of $y + 1 = \log_e x$

Shift 1 unit down

(iii) Transformation 3

$$y = f(x) \longrightarrow y = f(-x)$$

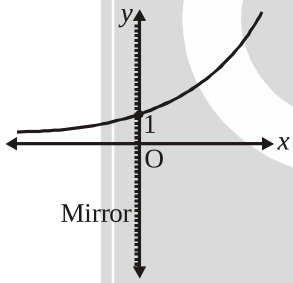
To plot $y = f(-x)$, Draw $y = f(x)$ first, then take the mirror image of $y = f(x)$ in the y-axis.

Example-4

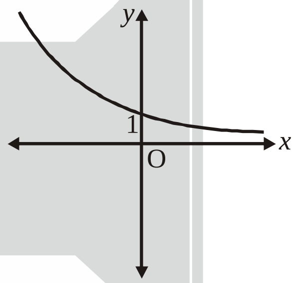
Sketch the graph of following function :

(a) $y - e^{-x} = 0$ **(b)** $y = \text{cosec}\,(-x)\ \forall\ x \in \left[-\frac{x}{2}, \frac{x}{2}\right]$

Soln. **(a)** $y = e^{-x} \xleftarrow[f(x)=f(-x)]{} y = e^{x}$



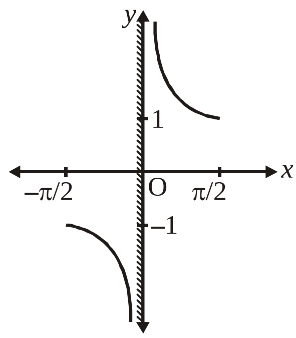
Graph of $y = e^x$



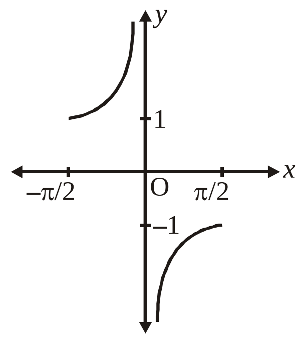
Graph of $y = e^{-x}$

Take mirror image about y-axis

(b) $y = \text{cosec}\,(-x) \xleftarrow[f(x)-f(-x)]{} y = \text{cosec}\,(x)$



Graph of $y = \text{cosec}\,(x)$



Graph of $y = \text{cosec}\,(-x)$

Take mirror image about y-axis

(iv) Transformation 4

$$y = f(x) \longrightarrow y = -f(x)$$

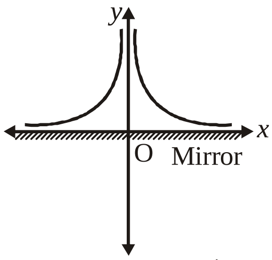
To plot $y = -f(x)$, Draw $y = f(x)$ first and then take the mirror image of $y = f(x)$ in x-axis.

Example-5

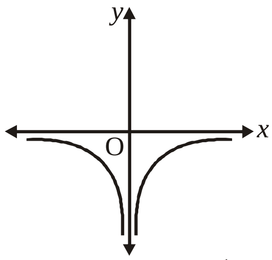
Sketch the graph of following functions :

(a) $x^2 y = -1$ **(b)** $y = \log 1/x$

Soln. **(a)** $y = -\dfrac{1}{x^2} \xleftarrow{f(x) = -f(x)} y = \dfrac{1}{x^2}$



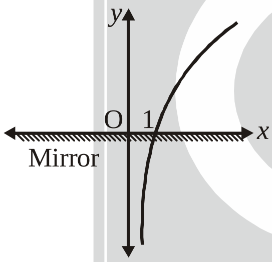
Graph of $y = \dfrac{1}{x^2}$



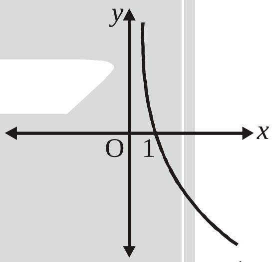
Graph of $y = -\dfrac{1}{x^2}$

Take mirror image about x-axis

(b) $y = \log \dfrac{1}{x} = -\log x \xleftarrow{f(x) = -f(x)} y = \log x$



Graph of $y = \log_e x$



Graph of $y = \log \dfrac{1}{x}$

Take mirror image about x-axis

(v) Transformation 5

$$y = f(x) \longrightarrow y = f(|x|)$$

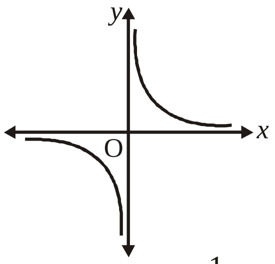
To plot $y = f(|x|)$, draw the graph of $y = f(x)$ first, then remove the portion of the graph in left half and after that take the mirror image of portion of the graph in right half in the y-axis. Also include the right portion of the graph of $y = f(x)$.

Example-6

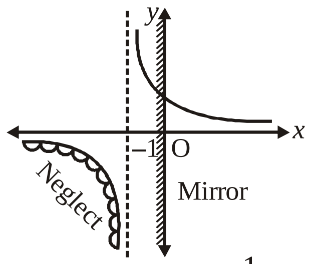
Sketch the graph of following functions :

(a) $y = \dfrac{1}{|x|+1}$ **(b)** $y = x^2 - 2|x| + 3$

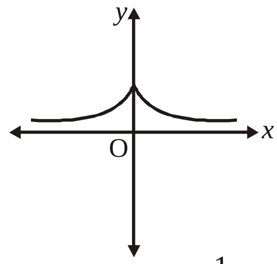
Soln. **(a)** $y = \dfrac{1}{|x|+1} \xleftarrow{f(x) = f(|x|)} y = \dfrac{1}{x+1} \xleftarrow{f(x) = f(x+a)} y = \dfrac{1}{x}$



Graph of $y = \frac{1}{x}$



Graph of $y = \frac{1}{x+1}$

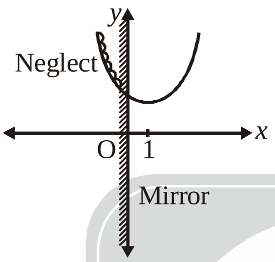


Graph of $y = \frac{1}{|x|+1}$

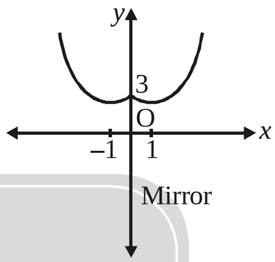
Shift 1 unit left

Reject left half and take reflection of right half in left half

(b) $y = x^2 - 2|x| + 3 \xleftarrow[f(x) = f(|x|)]{} y = x^2 - 2x + 3$



Graph of $y = x^2 - 2x + 3$



Graph of $y = x^2 - 2|x| + 3$

Reject left half and take reflection of right half in left half

(vi) Transformation 6

$$y = f(x) \longrightarrow y = |f(x)|$$

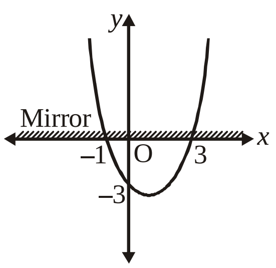
To plot $y = |f(x)|$, draw the curve $y = f(x)$, then take the mirror image of the lower portion of the curve (the curve below x-axis) in x-axis and then reject the lower part (or flip lower part into upper).

Example-7

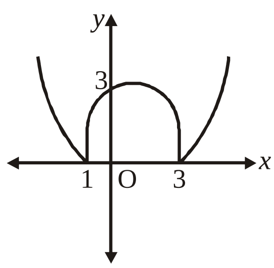
Draw the graph of the following curves :

(a) $y = |x^2 - 2x - 3|$ **(b)** $y = |\log x|$

Soln. **(a)** $y = |x^2 - 2x - 3| \xleftarrow[f(x) = |f(x)|]{} y = x^2 - 2x - 3$



Graph of $y = x^2 - 2x - 3$



Graph of $y = |x^2 - 2x - 3|$

Flip lower part into upper

(b) $y = |\log x| \xleftarrow[f(x) = |f(x)|]{} y = \log x$