Chapter 2

COUNTABLE OF SETS

Equivalent sets : Two sets A and B are said to be equivalent if $\exists$ a bijection. i.e., one-one and onto function, from a set A to a set B.
A is equivalent to B is written as $A \sim B$.

E.g. 1. $\mathbb{N} \sim \mathbb{Z}$

E.g. 2. $\mathbb{N} \sim \mathbb{Q}$

Finite set : A set P is said to be finite if $P = \phi$ or $P \sim N_m$ for some $m \in \mathbb{N}$, where $N_m := \{n \in \mathbb{N} : n \le m\}$

For example : Take $m = 5$, then $N_5 = \{n \in \mathbb{N} : n \le 5\} = \{1, 2, 3, 4, 5\}$

E.g. Take $P = \{1, 2, 3, 4, 5, ..., n\}$ is a finite set.

Infinite set : A set P which is not finite is called an infinite set. i.e., P is an infinite set if $P \ne \phi$ and $P \sim N_m, \forall\, m \in \mathbb{N}$.

E.g. (*i*) Take $P = \mathbb{N}$ is an infinite set

(*ii*) Take $P = \mathbb{Z}$ is an infinite set

Theorem:

(*i*) Suppose A is finite such that $B \subseteq A$, then B is finite.

(*ii*) Suppose A is an infinite set such that $A \subseteq B$, then B is infinite set.

(*iii*) If A and B are finite, then $A \cap B$ is also a finite.

(*iv*) If A and B are finite, then $A \cup B$ is also a finite.

(*v*) Every infinite set has a countable subset.

DEFINITIONS :

Denumerable set : A set, say P is said to be denumerable if $P \sim \mathbb{N}$.

Countable set : A set P is said to be countable if either P is finite or $P \sim \mathbb{N}$.

Uncountable set : A set which is not countable is called an uncountable.

E.g. 1. (*i*) Take $P = \mathbb{N}$ (set of natural numbers is a countable).

(*ii*) Take $P = \mathbb{Z}$ (set of integers) is a countable.

(*iii*) $\mathbb{Q}$ is a countable.

Examples of an uncountable sets :

(*i*) $Q^c = \mathbb{R} \setminus \mathbb{Q}$ is an uncountable set.

(*ii*) $\mathbb{R}$ is also an uncountable set.

Theorem : Every subset of a countable set is countable.

Proof : Let P be a countable set then P is either finite or denumerable.

Case-I : When P is finite, we know that every subset of a finite set is finite. Hence every subset of P is finite. So, by definition every subset of P is countable.

Case-II : When P is denumerable, since P is denumerable $\Rightarrow P \sim \mathbb{N}$, where $\mathbb{N}$ is a set of natural numbers.

Let $p = \{p_1, p_2, ..., p_n, ...\}$ and $B \subseteq P$.

Suppose B is finite then B is countable.

Suppose B is not finite. i.e., B is infinite, let n_1 be a least positive integer such that $p_{n_1} \in B$.

Since B is infinite, then $B \neq \{p_{n_1}\}$

Let n_2 be a least positive integer such that $n_2 > n_1$ and $p_{n_2} \in B$.

Let n_3 be a least positive integer such that $n_3 > n_2 > n_1$ such that $p_{n_3} \in B$.

Now continuing this way $B = \{p_{n_1}, p_{n_2}, ...\}$, where $n_1 < n_2 < n_3 < .....$

Consider a mapping $f : \mathbb{N} \to B$ defined by $f(r) = p_{n_r}, \forall r \in \mathbb{N}$.

Clearly, f is one-one and onto. Hence $\mathbb{N} \sim B$, i.e., B is denumerable

$\Rightarrow B$ is countable.

Note: Converse of the theorem need not be true.

Let $P = \mathbb{R}$ and $B = \mathbb{Q}$

Clearly, $\mathbb{Q} \subseteq \mathbb{R}$ i.e. $B \subseteq P$

Clearly, B is countable but P is uncountable.

Theorem : Suppose A and B are countable sets then

(*i*) Suppose A and B are countable set and

(*ii*) $A \cup B$ is also countable set.

Proof of (*i*) : Given that A and B countable sets we have to show that $A \cap B$ is countable.

Clearly, $A \cap B \subseteq A$ and $A \cap B \subseteq B$ (from set theory)

Since A and B are countable sets.

We know that every subset of a countable set is countable.

So, $A \cap B$ is a countable set.

Similarly, we can prove **(*ii*)**.

Theorem : Every super set of an uncountable set is uncountable.

i.e., suppose A is uncountable set such that $A \subseteq B$, then B is also uncountable set.

Proof : Given that A is an uncountable set and $A \subseteq B$.

Claim B is also an uncountable set.

Now we are going to prove by contradiction.

Suppose B is a countable set.

Since $A \subseteq B$ we know that every subset of a countable set is countable.

Hence, A is a countable set, which is a contradiction to the fact that A is an uncountable.

Hence, B is an uncountable set.

Theorem : Every infinite subset of a denumerable set is denumerable.

Proof : Suppose B is an infinite subset of a denumerable, say P.

Since, P is denumerable

$\Rightarrow P \sim \mathbb{N}$, i.e., $\mathbb{N} \sim P$

Let $P=\{p_1, p_2,....\}$ and $B\subseteq P$

Since B is infinite, let n_1 be a least positive integer such that $p_{n_1}\in B$.

Clearly, $B\neq\{p_{n_1}\}$, since B is infinite.

Since B is infinite, let n_2 be a least positive integer with $n_2>n_1$ such that $p_{n_2}\in B$.

Clearly, $B\neq\{p_{n_1}, p_{n_2}\}$

Since B is infinite set, let n_3 be a least positive integer with $n_3>n_2>n_1$ such that $p_{n_3}\in B$.

Now continuing this way, $B=\{p_{n_1}, p_{n_2}, p_{n_3},....\}$

Consider mapping $f:\mathbb{N}\to B$ defined by $f(q)=p_{n_q}, \forall\, q\in\mathbb{N}$
Clearly, f is one-one and onto.
Hence, $\mathbb{N}\sim B$
So, B is a denumerable set.

Theorem : Suppose A is finite set and B is countable set then $A\cup B$ is also a countable.

Theorem : Let $A=\{A_\alpha:\alpha\in\Lambda\}$, where Λ is an index set, be a family of countable set then $\bigcup_{\alpha\in\Lambda} A_\alpha$ is also a countable set.

Result : If $f:A\to B$ is a bijection (one-one and onto) and if B is uncountable then A is also uncountable.

E.g. 1. Consider a mapping $f:\left(-\frac{\pi}{2},\frac{\pi}{2}\right)\to\mathbb{R}$ defined by $f(x)=\tan x, \forall\, x\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

Clearly, f is one-one and onto, and $\mathbb{R}$ is an uncountable.

So, by the above result, $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is also an uncountable set.

E.g. 1. Consider a mapping $f:(0,1)\to\mathbb{R}$ defined by $f(x)=\begin{cases}\frac{2x-1}{x} & \text{if } 0<x<\frac{1}{2}\\ \frac{2x-1}{1-x} & \text{if } \frac{1}{2}\le x<1\end{cases}$.

Clearly, f is one-one and onto.
Since $\mathbb{R}$ is an uncountable.
So, by the above result (0, 1) is also an uncountable set.

Theorem : (*i*) Set of polynomial with rational coefficient is countable.
Theorem : (*ii*) Set of points of discontinuity of a monotonic function is countable.
Theorem : If A and B are countable sets then $A\times B$ is also a countable set.

E.g. Let $A=\mathbb{N}=B$

Clearly, $A=B=\mathbb{N}$ is a countable set.

So, by the above theorem, $\mathbb{N}\times\mathbb{N}$ is also a countable set.

Theorem : Suppose $A_1, A_2,...., A_n$ are countable sets then $A_1\times A_2\times A_3\times.....\times A_n$ is also a countable.

E.g. Let $A_1=A_2=....=A_n=\mathbb{Z}$ (set of integer numbers), then $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}\times....\times\mathbb{Z}$ is also a countable set.

Theorem : The set of irrational numbers is not countable. i.e., $Q^c = \mathbb{R}\backslash Q$ is not countable.

Proof : Suppose Q^c is a countable set.

We know that $\mathbb{R} = Q \cup Q^c$

Since Q is a countable set and Q^c is a countable set.

So, $Q \cup Q^c$ is also a countable set.

Hence, $\mathbb{R}$ is a countable set, which is a contradiction to the fact that $\mathbb{R}$ is an uncountable.

So, $Q^c = R\backslash Q$ is not a countable set. i.e., Q^c is uncountable set.

Theorem : Suppose $A_1, A_2, A_3, ..., A_n, ...$ are countable sets then $A_1 \times A_2 \times A_3 \times .... \times A_n \times ....$ is not a countable set.

Result : **(*i*)** A finite set is not equivalent to any of its proper subsets.

(*ii*) Every infinite set is equivalent to a proper subset of it self.

(*iii*) Set of polynomials with real coefficients is not countable (i.e. uncountable).

Ques. Does there exist an interval, which is countable ?

Ans. Yes, $(a, a), \forall\, a \in \mathbb{R}$

Clearly, (a, a) is a countable.

(*iv*) There does not exist onto function from a countable set to uncountable set.

(*v*) There does not exist one-one function from uncountable set to countable set.

(*vi*) Set of circles with center rationals and radius rational is countable.

(*vii*) Set of all points of the plane is uncountable.

Ques. Which of the following subsets of $\mathbb{R}^2$ are uncountable

(a) $\{(a, b) \in \mathbb{R}^2 : a \le b\}$

(b) $\{(a, b) \in \mathbb{R}^2 : a + b \in Q\}$

(c) $\{(a, b) \in \mathbb{R}^2 : a \cdot b \in \mathbb{Z}\}$

(d) $\{(a, b) \in \mathbb{R}^2 : a, b \in Q\}$

Ques. Let x denote the two point set $\{0, 1\}$ and write $x_j = \{0, 1\}, \forall\ j \in \mathbb{N}$. Let $y = \prod_{j=1}^{\infty} x_j$, then which is correct ?

(a) y is countable set

(b) cardinality of y is = card [0, 1]

(c) $\bigcup_{n=1}^{\infty}\left(\prod_{j=1}^{n} x_j\right)$ is countable

(d) y is uncountable

Ques. Prove or disprove that the set $A = \{\pm 1, \pm 4, \pm 9, \pm 16, ...\}$ is countable.

Result : If B is countable and A is uncountable then $A - B$ is uncountable where $B \subseteq A$.

Proof : Suppose A is uncountable and $B \subseteq A$ is a countable.

To show that $A - B$ is uncountable. Suppose $A - B$ is countable then $A = A - B \cup B$.

Since $A - B$ and B are countable sets.

$\Rightarrow$ $A - B \cup B$ is also a countable set.

$\Rightarrow$ A is a countable set, which is a contradiction to the fact that A is uncountable.

So, $A - B$ is uncountable.

Result : **(*i*)** Suppose A is countable and $f : A \to B$ onto then B is countable.

(*ii*) Let $f : A \to B$ be a function such that range of f is uncountable then A is also uncountable.

(*iii*) Every countable set is equivalent to a set of natural numbers $(\mathbb{N})$.

Theorem : Any open interval (a, b) is equivalent to any other open interval (c, d), where $a \neq b,\ c \neq d$.

Proof : Consider a mapping $f:(a, b) \to (c, d)$ defined by $f(x) = \dfrac{(d-c)}{(b-a)}(x-a)+c,\ \forall\ x \in (a, b)$.

Clearly, f is one-one and onto.

Hence, $(a, b) \sim (c, d)$.

Ques. Prove that $[0, 1] \sim \mathbb{R}$.

Result : **(*i*)** Every singleton set is countable.

(*ii*) The set of all positive rational numbers is countable.

(*iii*) The set of all negative rational numbers is countable.

(*iv*) Any closed interval $[a, b]$ is equivalent to any other closed interval $[c, d]$.

(*v*) Any two non-trivial intervals are equivalent.

(*vi*) Any open interval (a, b) is uncountable where $a \neq b$.

(*vii*) Any uncountable subset of $\mathbb{R}$ is equivalent to $\mathbb{R}$

(*viii*) Let $S = \{ f \mid f: A \to B \text{ is a function} \}$. Let $|A| = m$ and $|B| = n$ then $|S| = |B|^{|A|} = n^m$.

(*ix*) Every countable infinite set has cardinality χ_0 (aleph naught).

(*x*) Two finite set are equivalent iff they have same number of elements.

(*xi*) If A is equivalent to a subset of B and B is equivalent of A, then $A \sim B$.

(*xii*) Every countable set is equivalent to a subset of natural number $(\mathbb{N})$.

(*xiii*) Every set equivalent to finite set is finite.

(*xiv*) Every set equivalent to infinite set is infinite set.

(*xv*) Suppose $A \sim B$ and $B \sim C$ then $A \sim C$.

Cardinality of infinite set

(*i*) The cardinality of $\mathbb{N}$ is χ_0 (Aleph naught)

(*ii*) The cardinality of $\mathbb{R}$ is c (continum).

Note: **(*i*)** $\underbrace{\chi_0 + \chi_0 + \cdots + \chi_0}_{n\text{-times}} = \chi_0$

(*ii*) $n + \chi_0 = \chi_0,\ \forall\ n \in \mathbb{N}$

(*iii*) $\chi_0 + c = c$

(*iv*) $\underbrace{c + c + \cdots + c}_{n\text{-times}} = c$

(*v*) $n \cdot \chi_0 = \chi_0,\ \forall\ n \in \mathbb{N}$

(*vi*) $\underbrace{c \cdot c \cdot c \cdots c}_{n\text{-times}} = c$

(*vii*) $\chi_0 \cdot c = c$

(*viii*) $2^{\chi_0} = c > \chi_0$

(*ix*) ${\chi_0}^{\chi_0} = 2^{\chi_0} = c$

(*x*) $\chi_0^c = 2^c$

(*xi*) $\chi_0 < c$