Chapter 4

Phase And Group Velocity

4.1 WAVE TRAIN AND THE CONCEPT OF WAVE GROUP:

A wave train of finite length may be considered to be formed by the superposition of a theoretically infinite number of plane harmonic waves having continuously differing frequencies. For practical purpose the frequencies are limited within a finite range depending on the length of the wave train. The shorter is the length of the wave train the wider is the effective frequency range. In this sense a short enough wave train may be considered as a group of harmonic waves. It may be called a wave group or a wave packet.

4.2 PHASE VELOCITY, GROUP VELOCITY AND THEIR RELATIONSHIP:

When a monochromatic harmonic wave moves through a medium the velocity with which planes of constant phase move is called the phase velocity or wave velocity. A plane harmonic wave may be represented by $\psi(x,t) = \text{Re } Ae^{j(\omega t - kx)}$ ... (1)

For planes of constant phase, $\omega t - kx = \text{constant}$ ... (2)

Differentiating, $\frac{d}{dt}(\omega t - kx) = 0$ or, $\frac{dx}{dt} = \frac{\omega}{k} = c_p$ ... (3)

where c_p is called the phase velocity.

Practical waves are of finite duration and are not truly monochromatic. A short enough wave train may be considered as a wave group formed by the superposition of an infinite number of plane harmonic wave having slightly differing frequencies and phases. The wave group has a maximum of amplitude which falls off to zero not far from the maximum. If the medium is dispersive i.e., the phase velocity depends on frequency then the shape of the wave group changes as it travels through the medium. The maximum of the wave group travels with a velocity that is different from the velocities of the component waves. This velocity is known as group velocity (c_g).

4.3 RELATION BETWEEN PHASE VELOCITY AND GROUP VELOCITY:

The wave group is not situated at a fixed point but moves in space with a velocity, called group velocity, given by $c_g = \frac{dx}{dt} = \frac{d\omega}{dk}$ ... (4)

Since phase velocity $c_p = \omega / k \Rightarrow \omega = c_p k$ we can write

$$c_g = \frac{d}{dk}(c_p k) = c_p + k\frac{dc_p}{dk} = c_p + k\frac{dc_p}{d\lambda}\cdot\frac{d\lambda}{dk}$$

Now, $k = 2\pi / \lambda$ or, $d\lambda / dk = -2\pi / k^2$.

$$\Rightarrow \qquad c_g = c_p - \frac{2\pi}{k}.\frac{dc_p}{d\lambda} = c - \lambda\frac{dc_p}{d\lambda}$$

This is the required relation between phase velocity and group velocity for a dispersive medium. For a non-dispersive medium $dc_p/d\lambda = 0$. Then $\nu_g = c_p$ i.e, group velocity equals phase velocity.

SOLVED EXAMPLES

1. For gravity waves in a liquid the phase velocity c depends on the wavelength λ according to the formula $c_p = A\sqrt{\lambda}$, A being a constant. Show that the group velocity is half the phase velocity.

Soln. $c_p = A\sqrt{\lambda} \qquad \Rightarrow \frac{dc_p}{d\lambda} = \frac{1}{2}A\lambda^{-1/2}$

Therefore, $\qquad c_g = c_p - \lambda\frac{dc_p}{d\lambda} = c_p - \frac{1}{2}A\sqrt{\lambda} = c_p - \frac{c_p}{2} = \frac{1}{2}c_p$

2. The dispersion relation for microwaves in ionosphere is given by

$$\omega^2 = \omega_p^2 + c^2k^2$$

where c is the velocity of light in free space and ω_p is a constant depending on the electron density of the ionosphere. Show that the phase velocity c_p is greater than c. Does it violate the principle of relativity?

Soln. $\omega = \sqrt{\omega_p^2 + c^2k^2}$

Therefore, $\qquad$ Phase velocity $c_p = \frac{\omega}{k} = \sqrt{c^2 + \frac{\omega_p^2}{k^2}}$ which is greater than c.

Differentiating the dispersion relation with respect to k we get

$$2\omega\frac{d\omega}{dk} = c^2.2k \Rightarrow \frac{\omega}{k}.\frac{d\omega}{dk} = c^2$$

$$\Rightarrow \qquad c_pc_g = c^2 \qquad \left[\text{since } c_p > c,\ c_g < c\right]$$

It does not violate the principle of relativity because energy or signal is transmitted not with the phase velocity but with group velocity which is less than c.

3. Two harmonic waves represented by

$$\xi_1 = 3\cos(7t - 10x)\,m \text{ and } \xi_2 = 3\cos(5t - 8x)\,m$$

are superposed to form a wave group. Find the group velocity.

Soln. Angular frequencies are $\omega_1 = 7$ rad / s and 5 rad/s. Wave numbers are $k_1 = 10\,m^{-1}$ and $k_2 = 8\,m^{-1}$.

Therefore, $\qquad$ Group velocity $c_g = \frac{\Delta\omega}{\Delta k} = \frac{7-5}{10-8} = 1\,m/s$.

4. The phase velocity c_p of surface waves on a liquid of density ρ and surface tension S is given by

$$c_p^2 = \frac{g\lambda}{2\lambda} + \frac{2\pi S}{\rho\lambda}$$

where 'g' is the accleration due to gravity and λ is the wavelength of the wave.

(i) Obtain an expression for the group velocity of the wave.

(ii) Calculate the wavelength (λ_0) for which c is the minimum

(iii) Show that for $\lambda = \lambda_0$, group velocity $c_g = c_p$.

Soln. (i) Group velocity $c_g = \frac{d\omega}{dk} = \frac{d}{dk}(c_p k) = c_p + k\frac{dc_p}{dk}$

Since $k = \frac{2\pi}{\lambda},\ c_g = c_p - \lambda\frac{dc_p}{d\lambda}$

Now differentiating the given expression with respect to λ,

$$2c_p\frac{dc_p}{d\lambda} = \frac{g}{2\pi} - \frac{2\pi S}{\rho\lambda^2}$$

$$\Rightarrow \quad c_g = c_p - \frac{\lambda}{2c_p}\left(\frac{g}{2\pi} - \frac{2\pi S}{\rho\lambda^2}\right)$$

(ii) For c to be minimum, $\frac{dc}{d\lambda} = 0 \quad \Rightarrow \frac{g}{2\pi} - \frac{2\pi S}{\rho\lambda_0^2} = 0 \quad \Rightarrow \lambda_0 = 2\pi\sqrt{S/\rho g}$

(iii) For $\lambda = \lambda_0 \quad \Rightarrow c_g = c_p - \frac{\lambda_0}{2c_p}\left(\frac{g}{2\pi} - \frac{2\pi S}{\rho.4\pi^2(S/\rho g)}\right) = c_p - 0 = c_p$

5. A wave packet in a certain medium is constructed by superposing waves of frequency ω around $\omega_0 = 100$ and the corresponding wave-number k with $k_0 = 10$ as given in the table below

ω	k
81.00	9.0
90.25	9.5
100.00	10.0
110.25	10.5
121.00	11.0

Find the ratio v_g / g_p of the group velocity v_g and the phase velocity v_p.

(a) $\frac{1}{2}$ (b) 1 (c) $\frac{3}{2}$ (d) 2

[IIT-JAM : 2009]

Soln. **(d)** $v_p = \frac{\omega}{k} = \frac{100}{10} = 10$; $v_g^{(1)} = \frac{100 - 90.25}{10 - 9.5} = \frac{9.75}{0.5} = 19.5$

$v_g^{(2)} = \frac{110.25 - 100}{10.5 - 10} = \frac{10.25}{0.5} = 20.5$; $v_g = \frac{v_g^{(1)} + v_g^{(2)}}{2} = \frac{19.5 + 20.5}{2} = 20$

$\therefore \frac{v_g}{v_p} = \frac{20}{10} = 2$

6. For a wave in a medium the angular frequency 'ω' and the wave vector $\vec{k}$ are related by, $\omega^2 = \left(\omega_0^2 + c^2k^2\right)$, where ω_0 and c are constants. The product of group and phase velocities, i.e. $v_g v_p$ is:

(a) $0.25c^2$ (b) $0.4c^2$ (c) $0.5c^2$ (d) c^2

[IIT-JAM : 2010]

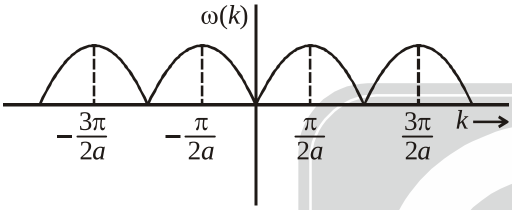
Soln. (d) $v_p = \frac{\omega}{k} = \frac{\sqrt{\omega_0^2 + c^2k^2}}{k}$; $v_g = \frac{d\omega}{dk} = \frac{c^2k}{\sqrt{\omega_0^2 + c^2k^2}}$ $\Rightarrow \boxed{v_g.v_p = c^2}$

7. The dispersion relation for a wave is given as $\omega(k) = C\left|\sin ka\right|$, where C is a constant. The group velocity of the wave vanishes at

(a) $k = 0$ (b) $k = \pm\pi/2a$ (c) $k = \pm\pi/2$ (d) $k = \pm 2\pi/a$

[IIT-JAM GP : 2008]

Soln. (b)



$\omega(k) = C\left|\sin ka\right|$

$v_g = \frac{d\omega}{dk} = 0$ at $k = \pm\frac{\pi}{2a}$

8. In a certain medium, the wave number k and the frequency ω are related by the dispersion relation $\omega^2 = c^2k^2(1+\alpha k^2)$, where c and α are constants. If v_g is the group velocity and v_p is the phase velocity, then

(a) $v_g = v_p$ (b) $\frac{v_g}{v_p} = \frac{(1+\alpha k^2)}{(1+2\alpha k^2)}$ (c) $v_g \cdot v_p = c^2$ (d) $v_g \cdot v_p = c^2(1+2\alpha k^2)$

[IIT-JAM GP : 2010]

Soln. (d)

$$\omega = ck(1+\alpha k^2)^{1/2}$$

$$v_g = \frac{d\omega}{dk} = c(1+\alpha k^2)^{1/2} + ck\frac{1}{2}(1+\alpha k^2)^{1/2}(2\alpha k)$$

$$= c(1+\alpha k^2)^{1/2} + \frac{\alpha ck^2}{(1+\alpha k^2)^{1/2}} = \frac{c(1+\alpha k^2) + \alpha ck^2}{(1+\alpha k^2)^{1/2}}$$

$$= \frac{c(1+2\alpha k^2)}{(1+\alpha k^2)^{1/2}}$$

Since $v_p = \frac{\omega}{k} = c(1+\alpha k^2)^{1/2}$

Therefore, $v_g \cdot v_p = c^2(1+2\alpha k^2)$

9. The dispersion relation for surface waves propagating in a fluid is given as: $\omega^2 = \alpha k + \beta k^3$, where α and β are constants with appropriate units. The phase velocity βV_p becomes equal to the group velocity V_g at $k = k_0$. The value of k_0 is

(a) $\sqrt{\frac{\alpha}{2\beta}}$ (b) $\sqrt{\frac{2\alpha}{\beta}}$ (c) $\sqrt{\frac{\alpha}{\beta}}$ (d) $\sqrt{\frac{3\alpha}{\beta}}$

[IIT-JAM GP : 2012]

Soln. **(c)** $\omega = \sqrt{\alpha k + \beta k^3}$

Phase velocity $V_p = \frac{\omega}{k} = \frac{\sqrt{\alpha k + \beta k^3}}{k}$

Group velocity $V_y = \frac{d\omega}{dk} = \frac{(\alpha + 3\beta k^2)}{2\sqrt{\alpha k + \beta k^3}}$

$V_p = V_g$

$\Rightarrow \frac{\sqrt{\alpha k + \beta k^3}}{k} = \frac{\alpha + 3\beta k^2}{2\sqrt{\alpha k + \beta k^3}} \Rightarrow 2(\alpha k + \beta k^3) = \alpha k + 3\beta k^3$

$\Rightarrow \beta k^3 - \alpha k = 0$

$k = \sqrt{\frac{\alpha}{\beta}}$

10. The angular frequency ω of deep water waves varies as the inverse square root of the wavelength λ, *i.e.*, $\omega(\lambda) \propto 1/\sqrt{\lambda}$. Which of the following is the relation between its group velocity v_g and phase velocity v_p?

(a) $v_g = v_p/2$ (b) $v_g = v_p$ (c) $v_g = 2v_p$ (d) $v_g = \lambda v_p$

[JNU : 2009]

Soln. **(a)** $\omega \propto \frac{1}{\sqrt{\lambda}}$; $\omega = \frac{a}{\sqrt{\lambda}}$, '$a$' is constant

$\therefore \quad \omega = \frac{a}{\sqrt{2\pi}}.\sqrt{k} \qquad \because \lambda = \frac{2\pi}{k}$

Or, $\omega^2 = \frac{a^2}{2\pi}k$

$\therefore \quad 2\omega\frac{d\omega}{dk} = \frac{a^2}{2\pi}; \qquad \frac{d\omega}{dk} = \frac{a^2}{4\pi}.\frac{1}{\omega} = \frac{a^2}{4\pi}.\frac{\sqrt{2\pi}}{a\sqrt{k}}$

$\Rightarrow \quad \frac{d\omega}{dk} = \frac{a}{2\sqrt{2\pi}}.\frac{1}{\sqrt{k}} = v_g$ and $\frac{\omega}{k} = \frac{a}{\sqrt{2\pi}}.\frac{\sqrt{k}}{k} = \frac{a}{\sqrt{2\pi}}.\frac{1}{\sqrt{k}} = v_p$

$\Rightarrow \quad v_g = \frac{1}{2}\left(\frac{a}{\sqrt{2\pi}}.\frac{1}{k}\right) = \frac{1}{2}v_p \Rightarrow \boxed{v_g = \frac{v_p}{2}}$