PAPER : IIT-JAM 2008
MATHEMATICS-MA

(CODE–A)

Q.1-Q.15: Only one option is correct for each question. Each question carries (+6) marks for correct answer and (–2) marks for incorrect answer.

1. The least positive integer *n*, such that $\begin{Bmatrix} \cos\pi/4 & \sin\pi/4 \\ -\sin\pi/4 & \cos\pi/4 \end{Bmatrix}^n$ is the identity matrix of order 2, is

(a) 4 (b) 8 (c) 12 (d) 16

2. Let $S = \{T : \mathbb{R}^3 \to \mathbb{R}^3; T$ is a linear transformation with $T(1,0,1) = (1,2,3)$ and $T(1,2,3) = (1,0,1)$. Then S is

(a) a singleton set

(b) a finite set containing more than one element

(c) a countable infinite set

(d) an uncountable set

3. Let $s_n = \int_0^1 \frac{n\,x^{n-1}}{(1+x)}dx$ for $n \geq 1$. Then as $n \to \infty$, the sequence $\{s_n\}$ tends to

(a) 0 (b) 1/2 (c) 1 (d) $+\infty$

4. The work done by the force $\vec{F} = 4y\hat{i} - 3xy\hat{j} + z^2\hat{k}$ in moving a particle over the circular path $x^2 + y^2 = 1, z = 0$ from $(1,0,0)$ to $(0,1,0)$ is

(a) $\pi + 1$ (b) $\pi - 1$ (c) $-\pi + 1$ (d) $-\pi - 1$

5. The set of all boundary points of $\mathbb{Q}$ in $\mathbb{R}$ is

(a) $\mathbb{R}$ (b) $\mathbb{R} \setminus \mathbb{Q}$ (c) $\mathbb{Q}$ (d) ϕ

6. Let $V = \left\{(x,y,z) \in \mathbb{R}^3 : \frac{1}{4} \leq x^2 + y^2 + z^2 \leq 1\right\}$ and $\vec{F} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\left(x^2+y^2+z^2\right)^2}$ for $(x,y,z) \in V$. Let $\hat{n}$ denote the outward unit normal vector to the boundary of V and S denote the part $\left\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = \frac{1}{4}\right\}$ of the boundary of *V*. Then $\iint_S \vec{F}.\hat{n}dS$ is equal to

(a) -8π (b) -4π (c) 4π (d) 8π

7. The set $U = \left\{x \in \mathbb{R} \mid \sin x = \frac{1}{2}\right\}$ is

(a) open (b) closed

(c) both open and closed (d) neither open nor closed

8. Let $f(x)=\int_0^x (x^2-t^2)g(t)dt$, where g is a real valued continuous function on $\mathbb{R}$. Then $f'(x)$ is equal to

(a) 0 (b) $x^3g(x)$ (c) $\int_0^x g(t)dt$ (d) $2x\int_0^x g(t)dt$

9. Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of the differential equation $y''+P(x)y'+Q(x)y=0$ where $P(x)$ and $Q(x)$ are continuous function on internal I. Then $y_3(x)=ay_1(x)+by_2(x)$ and $y_4(x)=cy_1(x)+dy_2(x)$ are linearly independent solutions of the given differential equation if

(a) $ad=bc$ (b) $ac=bd$ (c) $ad\neq bc$ (d) $ac\neq bd$

10. The set $R=\{f \mid f \text{ is a function from } \mathbb{Z} \text{ to } \mathbb{R}\}$ under the binary operations + and . defined as $(f+g)(n)=f(n)+g(n)$ and $(f.g)(n)=f(n)g(n)$ for all $n\in\mathbb{Z}$ forms a ring. Let $S_1=\{f\in R \mid f(-n)=f(n) \text{ for all } n\in\mathbb{Z}\}$ and $S_2=\{f\in R \mid f(0)=0\}$. Then

(a) S_1 and S_2 are both ideals in R (b) S_1 is an ideal in R while S_2 is not
(c) S_2 is an ideal in R while S_1 is not (d) neither S_1 nor S_2 is an ideal in R

11. Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be a linear transformation such that $T(1, 2, 3) = (1, 2, 3)$, $T(1, 5, 0) = (2, 10, 0)$ and $T(-1, 2, -1) = (-3, 6, -3)$. The dimension of the vector space spanned by all the eigenvectors of T is
(a) 0 (b) 1 (c) 2 (d) 3

12. Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers defined as $a_1=1$ and for $n\geq 1$, $a_{n+1}=a_n+(-1)^n 2^{-n}$, $b_n=\dfrac{2a_{n+1}-a_n}{a_n}$. Then

(a) $\{a_n\}$ converges to zero and $\{b_n\}$ is a Cauchy sequence
(b) $\{a_n\}$ converges to a non-zero number and $\{b_n\}$ is a Cauchy sequence
(c) $\{a_n\}$ converges to zero and $\{b_n\}$ is not a convergent sequence
(d) $\{a_n\}$ converges to a non-zero number and $\{b_n\}$ is not a convergent sequence

13. Let $f(-1,1)\to\mathbb{R}$ be defined as $f(x)=\dfrac{x^2}{1-\cos x}$ for $x\neq 0$ and $f(0)=2$. If $f(x)=\sum_{n=0}^{\infty} a_n x^n$ is the Taylor expansion of f for all x in $(-1,1)$, then $\sum_{n=0}^{\infty} a_{2n+1}$ is

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) 2

14. Let $y_1(x)$ and $y_2(x)$ be twice differentiable functions on a interval I satisfying the differential equations $\frac{dy_1}{dx} - y_1 - y_2 = e^x$ and $2\frac{dy_1}{dx} + \frac{dy_2}{dx} - 6y_1 = 0$. Then $y_1(x)$ is

(a) $C_1e^{-2x} + C_2e^{3x} - \frac{1}{4}e^x$ (b) $C_1e^{2x} + C_2e^{-3x} + \frac{1}{4}e^x$

(c) $C_1e^{2x} + C_2e^{-3x} - \frac{1}{4}e^x$ (d) $C_1e^{-2x} + C_2e^{3x} + \frac{1}{4}e^x$

15. Let G be a finite group and H be a normal subgroup of G of order 2. Then the order of the centre of G is

(a) 0 (b) 1

(c) an even integer ≥ 2 (d) an odd integer ≥ 3

16. **(a)** Let f and g be continuous functions on $\mathbb{R}$ such that $f(x) = \int_0^x g(t)\,dt$ and $g(x) = \int_x^0 f(t)\,dt + 1$.

Prove that $(f(x))^2 + (g(x))^2 = 1$ for all $x \in \mathbb{R}$. (6)

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that $f\,'$ is continuous on $\mathbb{R}$. Show that the series

$\sum_{n=1}^{\infty}\left(f\left(\frac{x}{2n}\right) - f\left(\frac{x}{2n+1}\right)\right)$ converges uniformly on [0, 1]. (9)

17. **(a)** Find the maxima, minima and saddle points, if any, for the function $f(x, y) = (y - x^2)(y - 2x^2)$ on $\mathbb{R}^2$. (6)

(b) Let $P(x) = a_0 + a_1x^2 + a_2x^4 + a_3x^6 + ... + a_nx^{2n}$, where $n \geq 1$ and $a_k > 0$ for $k = 0, 1, ... n$. Show that $P(x) - xP'(x) = 0$ has exactly two real roots. (9)

18. **(a)** Given that $y_1(x) = x$ is a solution of $(1 + x^2)y'' - 2xy' + 2y = 0, x > 0$, find a second linearly independent solution. (6)

(b) Solve $x^2y'' + xy' - y = 4x\log x, x > 0$. (9)

19. **(a)** Let ϕ be a differential function on [0, 1] satisfying $\phi'(x) \leq 1 + 3\phi(x)$ for all $x \in [0,1]$ with $\phi(0) = 0$. Show that $3\phi(x) \leq e^{3x} - 1$. (6)

(b) If $y_1(x) = x(1-2x)$, $y_2(x) = 2x(1-x)$ and $y_3(x) = x(e^x - 2x)$ are three solutions of a non-homogeneous linear differential equation $y'' + P(x)y' + Q(x)y = R(x)$, where $P(x), Q(x)$ and $R(x)$ are continuous functions on $[a, b]$ with a>0, then find its general solution. (9)

20. **(a)** Evaluate $\int_1^4 \int_0^1 \int_{2y}^2 \frac{\cos x^2}{\sqrt{z}} dx\, dy\, dz$. (6)

(b) Find the surface area of the portion of the cone $z^2 = x^2 + y^2$ that is inside the cylinder $z^2 = 2y$. (9)

21. **(a)** Using Green's theorem to evaluate the integral $\oint_C x^2 dx + (x + y^2) dy$, where C is the closed curve given by $y = 0, y = x$ and $y^2 = 2 - x$ in the first quadrant, oriented counter clockwise. (6)

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Use change of variables to prove that

$\iint_D f(x - y) dx\, dy = \int_{-1}^{1} f(u) du$ where $D = \{(x.y) \in R^2 : |x| + |y| \leq 1\}$. (9)

22. Using Gauss's divergence theorem, evaluate the integral $\iint_S \vec{F}.\hat{n} dS$, where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + 4yz\hat{k}$, S is the surface of the solid bounded by the sphere $x^2 + y^2 + z^2 = 10$ and the paraboloid $x^2 + y^2 = z - 2$, and $\hat{n}$ is the outward unit normal vector to *S*. (15)

23. **(a)** A square matrix M of order n with complex entries is called skew Hermitian if $M + \bar{M}^T = 0$ where 0 is the zero matrix of order n. Determine whether $V = \{M \mid M$ is a 2 × 2 skew Hermitian matrix} is a vector space over

(i) the field $\mathbb{R}$ and

(ii) the field $\mathbb{C}$ with usual operation of addition and scalar multiplication for matrices? (6)

(b) Let $V = \{P(x) \mid P(x)$ is a polynomial of degree $\leq n$ with real coefficients and $T : V \to \mathbb{R}^m$ be defined as $T(P(x)) = (P(1), P(2)........P(m))$. Show that T is linear and determine the nullity of T. (9)

24. Let G be the set of all 3×3 real matrices M such that $MM^T = M^T M = I_3$ and let $H = \{M \in G \mid \det M = 1\}$, where I_3 is the identity matrix of order 3. Then show that

(i) G is a group under matrix multiplication,

(ii) H is a normal subgroup of G,

(iii) $\phi : G \to \{-1, 1\}$ given by $\phi(M) = \det M$ is onto,

(iv) G/H is abelian. (15)

25. **(a)** Suppose that (R, +, .) is a ring having the property $a.b = c.a \Rightarrow b = c$, when $a \neq 0$. Then prove that $(R, +, .)$ is a commutative ring. (6)

(b) Let R be a commutative ring with identity. For $a_1, a_2, ..., a_n \in R$, the ideal generated by

$\{a_1, a_2, ..., a_n\}$ is given by

$$\langle a_1, a_2, ...a_n \rangle = \{r_1 a_1 + r_2 a_2 + ... + r_n a_n \mid r_i \in R, 1 \le i \le n\}.$$

Let $\mathbb{Z}[x]$ be the set of all polynomials with integer coefficients. Consider the ideal $I = \{f \in Z[x] \mid f(0) \text{ is an even integer}\}$. Prove that $I = \langle 2, x \rangle$ and that it is a maximal ideal. (9)

26. For a given positive integer $n > 1$, show that there exist subspaces $X_1, X_2, ...X_n$ of $\mathbb{R}^m$ for some integer $m>n$ and a linear transformation. $T : \mathbb{R}^m \to \mathbb{R}^m$ such that

- $\dim X_k = k, k = 1, 2, ...n$,
- for $i \ne j$, $X_i \cap X_j = \{\vec{0}\}$ where $\vec{0}$ is zero vector of $\mathbb{R}^m$,
- $T(X_k) = X_{k-1}, k = 1, 2, ...n,$ where $X_0 = \{\vec{0}\}$.

Also, find the rank of T. (15)

27. Let $f : (0, \infty) \to (0, \infty)$ be a continuously differentiable function and let $z = \dfrac{xy}{f(x^2 + y^2)}$ be defined for $xy \ne 0$.

(a) Prove that $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = \dfrac{x + y}{\left[f(x^2 + y^2)\right]^2} \left\{ f\left(x^2 + y^2\right) - 2xy \; f'\left(x^2 + y^2\right) \right\}$. (6)

(b) Further, if f is homogeneous of degree $\dfrac{1}{2}$, then verify that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$. (9)

28. Determine the interval of convergence of the power series $\sum_{n=1}^{\infty} n(2n-1)x^{2n}$ and show that its sum is $\dfrac{x^2(1+3x^2)}{(1-x^2)^3}$ at point x in its interval of convergence. (15)

29. **(a)** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as $f(x, y) = x^2 \cos\left(\dfrac{y}{x}\right)$ for $x \ne 0$ and $f(x, y) = 0$ for $x = 0$.

Compute $\dfrac{\partial f}{\partial x}$ at all points in $\mathbb{R}^2$ and show that it is continuous at the origin. (6)

(b) Let $f : (0,1) \to (0,\infty)$ be a uniformly continuous function. If $\{x_n\}$ is a Cauchy sequence in (0, 1), then prove that $\{f(x_n)\}$ is a Cauchy sequence in $(0,\infty)$.Hence deduce that for any two Cauchy sequences $\{x_n\}$ and $\{y_n\}$ in (0, 1), $\{|f(x_n) - f(y_n)|\}$ is a Cauchy sequence in $(0,\infty)$. (9)

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