SOLVED PAPER : IIT-JAM 2011

MATHEMATICS-MA

(CODE–A)

Q.1-Q.15: Only one option is correct for each question. Each question carries (+6) marks for correct answer and (–2) marks for incorrect answer.

1. Let $a_n = \sum_{k=1}^{n} \frac{n}{n^2+k}$, for $n \in \mathbb{N}$. Then the sequence $\{a_n\}$ is

(a) Convergent
(b) Bounded but not convergent
(c) Diverges to ∞
(d) Neither bounded nor diverges to ∞

2. The number of real roots of the equation $x^3 + x - 1 = 0$ is

(a) 0 (b) 1 (c) 2 (d) 3

3. The value of $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+kn}}$ is

(a) $2\left(\sqrt{2}-1\right)$ (b) $2\sqrt{2}-1$ (c) $2-\sqrt{2}$ (d) $\frac{1}{2}\left(\sqrt{2}-1\right)$

4. Let V be the region bounded by the planes $x=0, x=2, y=0, z=0$ and $y+z=1$. Then the value of integral $\iiint_V y\,dx\,dy\,dz$ is

(a) $\frac{1}{2}$ (b) $\frac{4}{3}$ (c) 1 (d) $\frac{1}{3}$

5. The solution $y(x)$ of the differential equation $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = 0$ satisfying the conditions $y(0) = 4, \frac{dy}{dx}(0) = 8$ is

(a) $4e^{2x}$
(b) $(16x+4)e^{-2x}$
(c) $4e^{-2x} + 16x$
(d) $4e^{-2x} + 16x\,e^{2x}$

6. If y^a is the integrating factor of the differential equation $2xydx - (3x^2 - y^2)dy = 0$, then the value of a is

(a) –4 (b) 4 (c) –1 (d) 1

7. Let $\vec{F} = ay\hat{i} + z\hat{j} + x\hat{k}$ and C be the positively oriented closed curve given by $x^2 + y^2 = 1, z = 0$. If $\oint_C \vec{F}.dr = \pi$, then the value of a is

(a) –1 (b) 0 (c) $\frac{1}{2}$ (d) 1

8. Consider the vector field $\vec{F}=(ax+y+a)\hat{i}+\hat{j}-(x+y)\hat{k}$, where a is a constant. If $\vec{F}.curl\,\vec{F}=0$, then the value of a is

(a) –1 (b) 0 (c) 1 (d) $\frac{3}{2}$

9. Let G denote the group of all 2×2 invertible matrices with entries from $\mathbb{R}$. Let
$$H_1=\{A\in G:\det(A)=1\} \text{ and } H_2=\{A\in G: A \text{ is upper triangular}\}.$$
Consider the following statements:
P : H_1 is a normal subgroup of G; Q : H_2 is normal subgroup of G
Then
(a) Both P and Q are true (b) P is true and Q is false
(c) P is false and Q is true (d) Both P and Q are false.

10. For $n\in\mathbb{N}$, let $n\mathbb{Z}=\{nk:k\in\mathbb{Z}\}$. Then the number of units of $\mathbb{Z}/11\mathbb{Z}$ and $\mathbb{Z}/12\mathbb{Z}$, respectively are
(a) 11, 12 (b) 10, 11 (c) 10, 4 (d) 10, 8

11. Let A be a 3 × 3 matrix with trace (A) = 3 and det (A) = 2. If 1 is an eigenvalue of A, then the eigen-values of the matrix $A^2 - 2I$ are
(a) $1, 2(i-1), -2(i+1)$ (b) $-1, 2(i-1), 2(i+1)$
(c) $1, 2(i+1), -2(i+1)$ (d) $-1, 2(i-1), -2(i+1)$

12. Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be a linear transformation, where $n\geq 2$. For $k\leq n$, let $E=\{v_1,v_2.....v_k\}\subseteq\mathbb{R}^n$ and $F=\{Tv_1,Tv_2.....Tv_k\}$.
Then
(a) If E is linearly independent, then F is linearly independent
(b) If F is linearly independent, then E is linearly independent
(c) If E is linearly independent, then F is linearly dependent
(d) If F is linearly independent, then E is linearly dependent

13. For $n\neq m$ let $T_1:\mathbb{R}^n\to\mathbb{R}^m$ and $T_2:\mathbb{R}^m\to\mathbb{R}^n$ be two linear transformations such that T_1T_2 is bijective. Then
(a) rank (T_1) = n and rank (T_2) = m (b) rank (T_1) = m and rank (T_2) = n
(c) rank (T_1) = n and rank (T_2) = n (d) rank (T_1) = m and rank (T_2) = m

14. The set of all x at which the power series $\sum_{n=1}^{\infty}\frac{n}{(2n+1)^2}(x-2)^{3n}$ converges is
(a) [–1, 1) (b) [–1, 1]
(c) [1, 3) (d) [1, 3]

15. Consider the following subsets of $\mathbb{R}$:
$$E=\left\{\frac{n}{n+1}:n\in\mathbb{N}\right\}, F=\left\{\frac{1}{1-x}:0\leq x<1\right\}$$

Then
(a) Both E and F are closed
(b) E is closed and F is NOT closed
(c) E is NOT closed and F is closed
(d) Neither E nor F is closed

16. **(a)** Let $\{a_n\}$ be a sequence of non-negative real numbers such that $\sum_{n=1}^{\infty} a_n$ converges, and let $\{k_n\}$ be a strictly increasing sequence of positive integers. Show that $\sum_{n=1}^{\infty} a_{k_n}$ also converges. (9)

(b) Suppose $f:[0,1]\to\mathbb{R}$ is differentiable and $f'(x)\le 1$ at every $x\in(0,1)$. If $f(0)=0$ and $f(1)=1$, show that $f(x)=x$ for all $x\in[0,1]$. (6)

17. **(a)** Suppose f is a real valued function defined on an open interval I and differentiable at every $x\in I$. If $[a,b]\subset I$ and $f'(a)<0<f'(b)$, then show that there exists $c\in(a,b)$ such that $f(c)=\min\limits_{a\le x\le b} f(x)$.

(b) Let $f:(a,b)\to\mathbb{R}$ be a twice differentiable function such that f'' is continuous at every point in (a,b). Prove that

$$\lim_{h\to 0}\frac{f(x+h)-2f(x)+f(x-h)}{h^2}=f''(x) \text{ for every } x\in(a,b).$$ (9 + 6)

18. Find all the critical points of the following function and check whether the function attains maximum or minimum at each of these points. (15)

$$u(x,y)=x^4+y^4-2x^2-2y^2+4xy,\ (x,y)\in\mathbb{R}^2.$$

19. **(a)** Let $\varphi:[a,b]\to\mathbb{R}$ be differentiable and $[c,d]=\{\varphi(x): a\le x\le b\}$, and let $f:[c,d]\to\mathbb{R}$ be continuous. Let $g:[a,b]\to\mathbb{R}$ be defined by $g(x)=\int_c^{\varphi(x)} f(t)\,dt$ for $x\in[a,b]$. Then show that g is differentiable and $g'(x)=f(\varphi(x))\varphi'(x)$ for all $x\in[a,b]$. (9)

(b) If $f:[0,1]\to\mathbb{R}$ is such that $\int_0^{\sin x} f(t)\,dt=\frac{\sqrt{3}}{2}x$ for all $x\in\mathbb{R}$, then find $f\left(\frac{1}{2}\right)$. (6)

20. Find the area of the surface of the solid bounded by the cone $z=3-\sqrt{x^2+y^2}$ and the paraboloid $z=1+x^2+y^2$. (15)

21. Obtain the general solution of each of the following differential equations:

(a) $y-x\dfrac{dy}{dx}=\dfrac{dy}{dx}y^2e^y$ (6)

(b) $\dfrac{dy}{dx}=\dfrac{x+2y+8}{2x+y+7}$ (9)

22. **(a)** Determine the value of $b > 1$ such that the differential equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + y = 0, 1 < x < b$$

satisfying the conditions $y(1) = 0 = y(b)$ has a nontrivial solution. (9)

(b) Find $v(x)$ such that $y(x) = e^{4x} v(x)$ is a particular solution of the differential equation

$$\frac{d^2 y}{dx^2} - 8\frac{dy}{dx} + 16y = (2x + 11x^{10} + 21x^{20})e^{4x} \quad (6)$$

23. **(a)** Change the order of integration in the double integral $\int_{-1}^{2}\left(\int_{-x}^{2-x^2} f(x,y)dy\right)dx$. (6)

(b) Let $\vec{F} = \left(x^2 - xy^2\right)\hat{i} + y^2\hat{j}$. Using Green's theorem, evaluate the line integral $\int_C \vec{F}.d\vec{r}$, where C is the positively oriented closed curved which is the boundary of the region enclosed by the x-axis and the semi-circle $y = \sqrt{1-x^2}$ in the upper half plane. (9)

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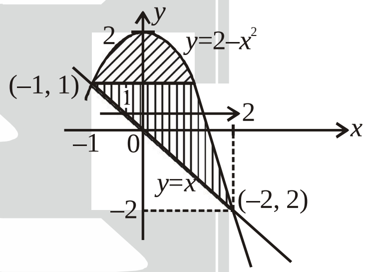
Soln. **(a)** $\int_{-1}^{2}\left(\int_{-x}^{2-x^2} f(x,y)dy\right)dx$



$= \iint_R f(x,y)dydx$ (R is as shown in the figure)

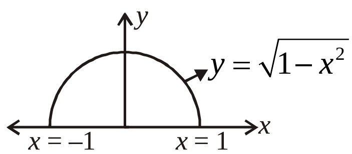
$= \int_{-2}^{1}\left(\int_{-y}^{\sqrt{2-y}} f(x,y)dxdy\right) + \int_{1}^{2}\left(\int_{-\sqrt{2-y}}^{\sqrt{2-y}} f(x,y)dx\right)dy$ (changing the order of integration).

(b) $\vec{F} = \left(x^2 - xy^2\right)\hat{i} + y^2\hat{j}$, $\vec{F}.d\,\vec{r} = (x^2 - xy^2)dx + y^2 dy$

$\therefore$ Using Green's theorem

$$\oint \vec{F}.d\,\vec{r} = \iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy = \iint\left(\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2 - xy^2)\right)dxdy$$

$$= \int\int_0^{\sqrt{1-x^2}} 2xydydx = \int_{-1}^{1} 2x.\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx = \int_{-1}^{1} x(1-x^2)dx$$



$$= \int_{-1}^{1}(x - x^3)dx = \left[\frac{x^2}{2} - \frac{x^4}{2}\right]_{-1}^{1} = \left[\frac{1}{2} - \frac{1}{4} - \frac{1}{2} + \frac{1}{4}\right] = 0$$

24. **(a)** If $\vec{F}=\left(x^2+y-4\right)\hat{i}+3xy\hat{j}+\left(2xz+z^2\right)\hat{k}$, then evaluate the surface integral $\iint_S\left(\nabla\times\vec{F}\right).\hat{n}dS$, where S is the surface of the cone $z=1-\sqrt{x^2+y^2}$ lying above the xy-plane and $\hat{n}$ is the unit normal to S making an acute angle with $\hat{k}$. (9)

(b) Show that the series $\sum_{n=1}^{\infty}\frac{x}{\sqrt{n}(1+n^p x^2)}$ converges uniformly on $\mathbb{R}$ for $p > 1$. (6)

25. **(a)** Find a value of c such that the following system of linear equations has no solution:

$$x+2y+3z=1,$$
$$3x+7y+cz=2,$$
$$2x+cy+12z=3.$$

(6)

(b) Let V be a vector space of all polynomials with real coefficients of degree at most n, where $n\geq 2$. Considering element of V as functions from $\mathbb{R}$ to $\mathbb{R}$, define $W=\left\{p\in V:\int_0^1 p(x)dx=0\right\}$.

Show that W is a subspace of V and dim(W) = n. (9)

26. **(a)** Let A be a 3 × 3 real matrix with det (A) = 6. Then find det (adj A). (6)

(b) Let v_1 and v_2 be non-zero vectors in $\mathbb{R}^n$, $n\geq 3$, such that v_2 is not a scalar multiple of v_1. Prove that there exists a linear transformation $T:\mathbb{R}^n\to\mathbb{R}^n$ such that $T^3=T$, $Tv_1=v_2$ and T has at least three distinct eigenvalues. (9)

27. **(a)** If E is a subset of $\mathbb{R}$ that does not contain any of its limit points, then prove that E is a countable set. (9)

(b) Let $f:(a,b)\to\mathbb{R}$ be a continuous function. If f is uniformly continuous, then prove that there exists a continuous function $g:[a,b]\to\mathbb{R}$ such that $g(x)=f(x)$ for all $x\in(a,b)$. (6)

28. **(a)** On $\mathbb{R}^3$, define a binary operation * as follows: For $(x,y,t),(x',y',t')$ in $\mathbb{R}^3$,

$$(x,y,t)*(x',y',t')=\left(x+x',\ y+y',\ t+t'+\frac{1}{2}(x'y-xy')\right).$$

Then show that $\left(\mathbb{R}^3,*\right)$ is a group, and find its centre. (9)

(b) $k\in\mathbb{N}$ let $k\mathbb{Z}=\{kn:n\in\mathbb{Z}\}$. For any $m,n\in\mathbb{N}$ show that $I=m\mathbb{Z}\cap n\mathbb{Z}$ is an ideal of $\mathbb{Z}$. Further, find the generators of I.
(6)

29. Let G be a group of order p^2, where p is a prime number. Let $x \in G$. Prove that $\{y \in G : xy = yx\} = G$.

(15)

***** END *****