PAPER : IIT-JAM 2012
MATHEMATICS-MA

(CODE–A)

Q.1-Q.15: Only one option is correct for each question. Each question carries (+6) marks for correct answer and (–2) marks for incorrect answer.

1. Let $\{x_n\}$ be the sequence $+\sqrt{1}, -\sqrt{1}, +\sqrt{2}, -\sqrt{2}, +\sqrt{3}, -\sqrt{3}, +\sqrt{4}, -\sqrt{4}, ...$ If

$$y_n = \frac{x_1 + x_2 + ... + x_n}{n} \text{ for all } n \in \mathbb{N},$$

then the sequence $\{y_n\}$ is

(a) monotonic
(b) NOT bounded
(c) bounded but NOT convergent
(d) convergent

2. The number of distinct real roots of the equation $x^9 + x^7 + x^5 + x^3 + x + 1 = 0$ is

(a) 1 (b) 3 (c) 5 (d) 9

3. If $f : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$f(x, y) = \begin{cases} \dfrac{x^3}{x^2 + y^4} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$$

Then

(a) $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$
(b) $f_x(0, 0) = 1$ and $f_y(0, 0) = 0$
(c) $f_x(0, 0) = 0$ and $f_y(0, 0) = 1$
(d) $f_x(0, 0) = 1$ and $f_y(0, 0) = 1$

4. The value of $\int_{z=0}^{1} \int_{y=0}^{z} \int_{x=0}^{y} xy^2 z^3 \, dx\, dy\, dz$ is

(a) $\frac{1}{90}$ (b) $\frac{1}{50}$ (c) $\frac{1}{45}$ (d) $\frac{1}{10}$

5. The differential equation $(1 + x^2 y^3 + \alpha x^2 y^2)dx + (2 + x^3 y^2 + x^3 y)dy = 0$ is exact if α equals

(a) $\frac{1}{2}$ (b) $\frac{3}{2}$ (c) 2 (d) 3

6. An integrating factor for the differential equation $(2xy + 3x^2 y + 6y^3)dx + (x^2 + 6y^2)dy = 0$ is

(a) x^3 (b) y^3 (c) e^{3x} (d) e^{3y}

7. For $c > 0$, if $a\hat{i} + b\hat{j} + c\hat{k}$ is the unit normal vector at $\left(1, 1, \sqrt{2}\right)$ to the cone $z = \sqrt{x^2 + y^2}$, then

(a) $a^2 + b^2 - c^2 = 0$
(b) $a^2 - b^2 + c^2 = 0$
(c) $-a^2 + b^2 + c^2 = 0$
(d) $a^2 + b^2 + c^2 = 0$

8. Consider the quotient group $\mathbb{Q}/\mathbb{Z}$ of the additive group of rational numbers. The order of the element $\frac{2}{3}+\mathbb{Z}$ in $\mathbb{Q}/\mathbb{Z}$ is

(a) 2 (b) 3 (c) 5 (d) 6

9. Which one of the following is TRUE?

(a) The characteristic of the ring $6\mathbb{Z}$ is 6

(b) The ring $6\mathbb{Z}$ has a zero divisor

(c) The characteristic of the ring $(\mathbb{Z}/6\mathbb{Z})\times 6\mathbb{Z}$ is zero

(d) The ring $6\mathbb{Z}\times 6\mathbb{Z}$ is an integral domain.

10. Let *W* be a vector space over $\mathbb{R}$ and let $T:\mathbb{R}^6 \to W$ be a linear transformation such that $S=\{Te_2, Te_4, Te_6\}$ spans *W*. Which one of the following must be TRUE?

(a) S is a basis of W

(b) $T(\mathbb{R}^6)\neq W$

(c) $\{Te_1, Te_3, Te_5\}$ spans W

(d) ker (*T*) contains more than one element

11. Consider the following subspace of $\mathbb{R}^3$

$W=\{(x,y,z)\in\mathbb{R}^3 \mid 2x+2y+z=0, 3x+3y-2z=0, x+y-3z=0\}$.

The dimension of *W* is

(a) 0 (b) 1 (c) 2 (d) 3

12. Let *P* be a 4 × 4 matrix whose determinant is 10. The determinant of the matrix –3*P* is

(a) –810 (b) –30 (c) 30 (d) 810

13. If the power series $\sum_{n=0}^{\infty} a_n x^n$ converges for $x=3$, then the series $\sum_{n=0}^{\infty} a_n x^n$

(a) converges absolutely for $x=-2$

(b) converges but not absolutely for $x=-1$

(c) converges but not absolutely for $x=1$

(d) diverges for $x=-2$

14. If $Y=\left\{\frac{x}{1+|x|} \,\middle|\, x\in\mathbb{R}\right\}$, then the set of all limit points of *Y* is

(a) (–1, 1) (b) (–1, 1] (c) [0, 1] (d) [–1, 1]

15. If *C* is a smooth curve in $\mathbb{R}^3$ from (0, 0, 0) to (2, 1, –1), then the value of

$$\int_C (2xy+z)dx+(z+x^2)dy+(x+y)dz$$

is

(a) –1 (b) 0 (c) 1 (d) 2

16. **(a)** Examine whether the following series is convergent:

$$\sum_{n=1}^{\infty}\frac{n!}{1\cdot 3\cdot 5\cdots(2n-1)}. \qquad (6)$$

(b) For each $x\in\mathbb{R}$, let $[x]$ denote the greatest integer less than or equal to *x*. Further, for a fixed $\beta\in(0,1)$, define $a_n=\frac{1}{n}[n\beta]+n^2\beta^n$ for all $n\in\mathbb{N}$. Show that the sequence $\{a_n\}$ converges to β.

(9)

17. **(a)** Evaluate $\lim_{x\to 0}\frac{\int_0^{x^2}\sqrt{4+t^3}\,dt}{x^2}$. (6)

(b) For $a, b\in\mathbb{R}$ with $a<b$, let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b). Further assume that the graph of f intersects the straight line segment joining the points $(a,\ f(a))$ and $(b,\ f(b))$ at a point $(c,\ f(c))$ for $a<c<b$. Show that there exists a real number $\xi\in(a,b)$ such that $f''(\xi)=0$. (9)

18. **(a)** Show that the point (0, 0) is neither a point of local minimum nor a point of local maximum for the function $f:\mathbb{R}^2\to\mathbb{R}$ given by $f(x,y)=3x^4-4x^2y+y^2$ for $(x,y)\in\mathbb{R}^2$. (6)

(b) Find all the critical points of the function $f:\mathbb{R}^2\to\mathbb{R}$ given by $f(x,y)=x^3+y^3-3x-12y+40$ for $(x,y)\in\mathbb{R}^2$. Also, examine whether the function f attains a local maximum or a local minimum at each of these critical points. (9)

19. **(a)** Evaluate $\int\limits_{x=0}^{4}\int\limits_{y=\sqrt{4-x}}^{2} e^{y^3}\,dy\,dx$. (6)

(b) Using multiple integral, find the volume of the solid region in $\mathbb{R}^3$ bounded above by the hemisphere $z=1+\sqrt{1-x^2-y^2}$ and bounded below by the cone $z=\sqrt{x^2+y^2}$. (9)

20. Find the area of the portion of the surface $z=x^2-y^2$ in $\mathbb{R}^3$ which lies inside the solid cylinder $x^2+y^2\le 1$. (15)

21. Let $y(x)$ be the solution of the differential equation $\frac{d^2y}{dx^2}-y=0$ such that $y(0)=2$ and $y'(0)=2\alpha$. Find all values of $\alpha\in[0,1)$ such that the infimum of the set $\{y(x)\mid x\in\mathbb{R}\}$ is greater than or equal to 1. (15)

22. **(a)** Assume that $y_1(x)=x$ and $y_2(x)=x^3$ are two linearly independent solutions of the homogeneous differential equation $x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+3y=0$. Using the method of variation of parameters, find a particular solution of the differential equation

$$x^2\frac{d^2y}{dx^2}-3x\frac{dy}{dx}+3y=x^5$$ (6)

(b) Solve the differential equation $\frac{dy}{dx}+\frac{5y}{6x}=\frac{5x^4}{y^5}$ subject to the condition $y(1)=1$ (9)

23. **(a)** Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector field in $\mathbb{R}^3$ and let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that $\vec{\nabla} \times \left\{ f\left(|\vec{r}|\right)\vec{r} \right\} = \vec{0}$ for $\vec{r} \neq \vec{0}$. (6)

(b) Let *W* be the region inside the solid cylinder $x^2 + y^2 \leq 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$. Let *S* be the boundary of *W*. Using Gauss's divergence theorem, evaluate $\iint_S \vec{F} \cdot \hat{n}\, dS$, where $\vec{F} = (x^2 + y^2 - 4)\hat{i} + (3xy)\hat{j} + (2xz + z^2)\hat{k}$ and $\hat{n}$ is the outward unit normal vector to *S*. (9)

24. **(a)** Let *G* be a finite group whose order is not divisible by 3. Show that for every $g \in G$, there exists an $h \in G$ such that $g = h^3$. (6)

(b) Let *A* be the group of all rational numbers under addition, *B* be the group of all non-zero rational numbers under multiplication and *C* the group of all positive rational numbers under multiplication. Show that no two of the groups *A*, *B* and *C* are isomorphic. (9)

25. **(a)** Let *I* be an ideal of a commutative ring *R*. Define

$$A = \left\{ r \in R \mid r^n \in I \text{ for some } n \in \mathbb{N} \right\}.$$

Show that *A* is an ideal of *R*. (6)

(b) Let F be a field. For each $p(x) \in F[x]$ (the polynomial ring in x over F) define $\varphi : F[x] \to F \times F$ by $\varphi\left(p(x)\right) = \left(p(0), p(1)\right)$.

(i) Prove that φ is a ring homomorphism

(ii) Prove that the quotient ring $F[x]/\left(x^2 - x\right)$ is isomorphic to the ring $F \times F$. (9)

26. **(a)** Let *P*, *D* and *A* be real square matrices of the same order such that *P* is invertible. *D* is diagonal and $D = PAP^{-1}$. If $A^n = 0$ for some $n \in \mathbb{N}$, then show that $A = 0$. (6)

(b) Let $T : V \to W$ be a linear transformation of vector spaces. Prove the following:

(1) If $\{v_1, v_2, ..., v_k\}$ spans *V* and *T* is onto, then $\{Tv_1, Tv_2.....Tv_k\}$ spans *W*.

(2) If $\{v_1, v_2, ..., v_k\}$ is linearly independent in *V* and *T* is one-one, then $\{Tv_1, Tv_2.....Tv_k\}$ is linearly independent in *W*.

(3) If $\{v_1, v_2.....v_k\}$ is a basis of *V*, and *T* is bijective, then $\{Tv_1, Tv_2.....Tv_k\}$ is a basis of *W*. (9)

27. **(a)** Let $\{v_1, v_2, v_3\}$ be a basis of vector space *V* over $\mathbb{R}$. Let $T : V \to V$ be the linear transformation determined by $Tv_1 = v_1, Tv_2 = v_2 - v_3$ and $Tv_3 = v_2 + 2v_3$. Find the matrix of the transformation *T* with $\{v_1 + v_2, v_1 - v_2, v_3\}$ as the basis of both the domain and co-domain of *T*. (6)

(b) Let *W* be a three dimensional vector space over $\mathbb{R}$ and let $S : W \to W$ be a linear transformation. Further, assume that every non-zero vector of *W* is an eigen-vector of *S*. Prove that there exists an $\alpha \in \mathbb{R}$, such that $S = \alpha I$, where $I : W \to W$ is the identity transformation. (9)

28. **(a)** Show that the function $f:\mathbb{R}\to\mathbb{R}$, defined by $f(x)=x^2$ for $x\in\mathbb{R}$, is not uniformly continuous. (6)

(b) For each $n\in\mathbb{N}$, let $f_n:\mathbb{R}\to\mathbb{R}$ be a uniformly continuous function. If the sequence $\{f_n\}$ converges uniformly on $\mathbb{R}$ to a function $f:\mathbb{R}\to\mathbb{R}$, then show that f is uniformly continuous. (9)

29. **(a)** Let A be a nonempty bounded subset of $\mathbb{R}$. Show that $\{x\in\mathbb{R}\,|\,x\geq a \text{ for all } a\in A\}$ is closed subset of $\mathbb{R}$. (6)

(b) Let $\{x_n\}$ be a sequence in $\mathbb{R}$ such that $|x_{n+1}-x_n|<\frac{1}{n^2}$ for all $n\in\mathbb{N}$. Show that the sequence $\{x_n\}$ is convergent. (9)

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