PAPER : IIT-JAM 2013
MATHEMATICS-MA

(CODE–A)

(OBJECTIVE QUESTIONS)

Q.1-Q.10: Only one option is correct for each question. Each question carries (+2) marks for correct answer and (–0.5) marks for incorrect answer.

1. Let $A=\begin{bmatrix}1 & 1 & 1\\ 3 & -1 & 1\\ 1 & 5 & 3\end{bmatrix}$ and V be the vector space of all $X \in \mathbb{R}^3$ such that $AX = 0$. Then dim(V) is

(a) 0 (b) 1 (c) 2 (d) 3

2. The value of n for which the divergence of the function $\vec{F}=\dfrac{\vec{r}}{|\vec{r}|^n}, \vec{r}=x\hat{i}+y\hat{j}+z\hat{k}, |\vec{r}|\neq 0$, vanishes is

(a) 1 (b) –1 (c) 3 (d) –3

3. Let A and B be subsets of $\mathbb{R}$. Which of the following is NOT necessarily true?

(a) $(A\cap B)^0 \subseteq A^0 \cap B^0$ (b) $A^0 \cup B^0 \subseteq (A\cup B)^0$

(c) $\bar{A}\cup\bar{B} \subseteq \overline{A\cup B}$ (d) $\bar{A}\cap\bar{B} \subseteq \overline{A\cap B}$

4. Let $[x]$ denote the greatest integer function of x. The value of α for which the function

$$f(x)=\begin{cases}\dfrac{\sin[-x^2]}{[-x^2]}, & x\neq 0\\ \alpha, & x=0\end{cases}$$

is continuous at $x = 0$ is

(a) 0 (b) $\sin(-1)$ (c) sin 1 (d) 1

5. Let the function $f(x)$ be defined by

$$f(x)=\begin{cases}e^x, & x \text{ is rational}\\ e^{1-x}, & x \text{ is irrational}\end{cases}$$

for x in (0, 1). Then

(a) f is continuous at every point in (0, 1)

(b) f is discontinuous at every point in (0, 1)

(c) f is discontinuous only at one point in (0, 1)

(d) f is continuous only at one point in (0, 1)

6. The value of the integral $\iint_D \sqrt{x^2+y^2}\,dx\,dy, D=\{(x,y)\in\mathbb{R}^2; x\le x^2+y^2\le 2x\}$ is

(a) 0 (b) 7/9 (c) 14/9 (d) 28/9

7. Let $x_n=\left(1-\frac{1}{3}\right)^2\left(1-\frac{1}{6}\right)^2\left(1-\frac{1}{10}\right)^2\cdots\left(1-\frac{1}{\frac{n(n+1)}{2}}\right)^2$, $n\ge 2$. Then $\lim_{n\to\infty} x_n$ is

(a) $\frac{1}{3}$ (b) $\frac{1}{9}$ (c) $\frac{1}{81}$ (d) 0

8. Let p be a prime number. Let G be the group of all 2×2 matrices over $\mathbb{Z}_p$ with determinant 1 under matrix multiplication. Then the order of G is

(a) $(p-1)p(p+1)$ (b) $p^2(p-1)$

(c) p^3 (d) $p^2(p-1)+p$

9. Let V be the vector space of all 2 × 2 matrices over $\mathbb{R}$. Consider the subspaces

$W_1=\left\{\begin{pmatrix} a & -a \\ c & d\end{pmatrix}; a,c,d\in\mathbb{R}\right\}$ and $W_2=\left\{\begin{pmatrix} a & b \\ -a & d\end{pmatrix}: a,b,d\in\mathbb{R}\right\}$.

If $m=\dim(W_1\cap W_2)$ and $n=\dim(W_1+W_2)$, then the pair (m, n) is

(a) (2, 3) (b) (2, 4) (c) (3, 4) (d) (1, 3)

10. Let P_n be the real vector space of all polynomials of degree at most n. Let $D: P_n\to P_{n-1}$ and $T: P_n\to P_{n+1}$ be the linear transformations defined by

$D(a_0+a_1x+a_2x^2+.....+a_nx^n)=a_1+2a_2x+.....+na_nx^{n-1}$,

$T(a_0+a_1x+a_2x^2+.....+a_nx^n)=a_0x+a_1x^2+.....+a_nx^{n+1}$,

respectively. If A is the matrix representation of the transformation $DT-TD: P_n\to P_n$ with respect to the standard basis of P_n, then the trace of A is

(a) $-n$ (b) n (c) $n+1$ (d) $-(n+1)$

(FILL IN THE BLANKS)

Q.11-Q.20: Each question carries (+3) marks for correct answer. There is no negative marks.

11. The equation of the curve satisfying $\sin y\frac{dy}{dx}=\cos y(1-x\cos y)$ and passing through the origin is_______.

12. Let f be a continuously differentiable function such that $\int_0^{2x^2} f(t)dt=e^{\cos x^2}$ for all $x\in(0,\infty)$. The value of $f'(\pi)$ is_______.

13. Let $u=\dfrac{y^2-x^2}{x^2y^2}, v=\dfrac{z^2-y^2}{y^2z^2}$ for $x\neq 0, y\neq 0, z\neq 0$. Let $w=f(u,v)$, where f is a real valued function defined on $\mathbb{R}^2$ having continuous first order partial derivatives. The value of $x^3\dfrac{\partial w}{\partial x}+y^3\dfrac{\partial w}{\partial y}+z^3\dfrac{\partial w}{\partial z}$ at the point (1, 2, 3) is_______.

14. The set of points at which the function $f(x,y)=x^4+y^4-x^2-y^2+1, (x,y)\in\mathbb{R}^2$ attains local maximum is_______.

15. Let C be the boundary of the region in the first quadrant bounded by $y=1-x^2, x=0$ and $y=0$, oriented counter-clockwise. The value of $\oint_C (xy^2dx-x^2y\,dy)$ is_______.

16. Let $f(x)=\begin{cases}0, & -1\leq x\leq 0\\ x^4, & 0<x\leq 1\end{cases}$. If $f(x)=\sum_{k=0}^{n}\dfrac{f^{(k)}(0)}{k!}x^k+\dfrac{f^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$ is the Taylor's formula for f about $x = 0$ with maximum possible value of n, then the value of ξ for $0<x\leq 1$ is_______.

17. Let $\vec{F}=2z\hat{i}+4x\hat{j}+5y\hat{k}$, and let C be the curve of intersection of the plane $z=x+4$ and the cylinder $x^2+y^2=4$, oriented counter-clockwise. The value of $\oint_C \vec{F}\cdot d\vec{r}$ is_______.

18. Let f and g be the functions from $\mathbb{R}\setminus\{0,1\}$ to $\mathbb{R}$ defined by $f(x)=\dfrac{1}{x}$ and $g(x)=\dfrac{x-1}{x}$ for $x\in\mathbb{R}\setminus\{0,1\}$. The smallest group of functions from $\mathbb{R}\setminus\{0,1\}$ to $\mathbb{R}$ containing f and g under composition of functions is isomorphic to_______.

19. The orthogonal trajectory of the family of curves $\dfrac{x^2}{2}+y^2=c$, which passes through (1, 1) is_______.

20. The function to which the power series $\sum_{n=1}^{\infty}(-1)^{n+1}n\,x^{2n-2}$ converges is_______.

(DESCRIPTIVE QUESTIONS)

Q.21-Q.30: Each question carries (+5) marks.

21. Let $0 < a \le 1$, $s_1 = \frac{a}{2}$ and for $n \in \mathbb{N}$, let $s_{n+1} = \frac{1}{2}\left(s_n^2 + a\right)$. Show that the sequence $\{s_n\}$ is convergent, and find its limit.

22. Evaluate $\int_{1/4}^{1} \int_{\sqrt{x-x^2}}^{\sqrt{x}} \frac{x^2 - y^2}{x^2}\, dy\, dx$ by changing the order of integration.

23. Find the general solution of the differential equation

$$x^2 \frac{d^3 y}{dx^3} + x\frac{d^2 y}{dx^2} - 6\frac{dy}{dx} + 6\frac{y}{x} = \frac{x\ln x + 1}{x^2}, x > 0$$

24. Let S_1 be the hemisphere $x^2 + y^2 + z^2 = 1, z > 0$ and S_2 be the closed disc $x^2 + y^2 \le 1$ in the xy-plane. Using Gauss' divergence theorem, evaluate $\iint_S \vec{F} \cdot d\vec{S}$, where

$$\vec{F} = z^2 x\hat{i} + \left(\frac{y^3}{3} + \tan z\right)\hat{j} + (x^2 z + y^2)\hat{k} \text{ and } S = S_1 \cup S_2.$$

Also evaluate $\iint_{S_1} \vec{F} \cdot d\vec{S}$.

25. Let $f(x, y) = \begin{cases} \frac{2\left(x^3 + y^3\right)}{x^2 + 2y}, & (x, y) \ne (0, 0) \\ 0, & (x, y) = (0, 0). \end{cases}$

Show that the first order partial derivatives of f with respect to x and y exist at (0, 0). Also show that f is not continuous at (0, 0).

26. Let A be an $n \times n$ diagonal matrix with characteristic polynomial $(x - a)^p (x - b)^q$, where a and b are distinct real numbers. Let V be the real vector space of all $n \times n$ matrices B such that $AB = BA$. Determine the dimension of V.

27. Let A be an $n \times n$ real symmetric matrix with n distinct eigenvalues. Prove that there exists an orthogonal matrix P such that $AP = PD$, where D is a real diagonal matrix.

28. Let K be a compact subset of of $\mathbb{R}$ with nonempty interior. Prove that K is of the form [a, b] or of the form $[a, b] \setminus \cup I_n$, where $\{I_n\}$ is a countable disjoint family of open intervals with end points in K.

29. Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that f is differentiable in (a, c) and (c, b), $a < c < b$. If $\lim_{x\to c} f'(x)$ exists, then prove that f is differentiable at c and $f'(c) = \lim_{x\to c} f'(x)$.

30. Let G be a finite group, and let φ be an automorphism of G such that $\varphi(x) = x$ if and only if $x = e$, where e is the identity element in G. Prove that every $g \in G$ can be represented as $g = x^{-1}\varphi(x)$ for some $x \in G$. Moreover, if $\varphi(\varphi(x)) = x$ for every $x \in G$, then show that G is abelian.

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