PAPER : IIT-JAM 2014
MATHEMATICS-MA

(CODE–A)

PART-I
(OBJECTIVE QUESTIONS)

Q.1-Q10: Only one option is correct. Each question carries (+1) mark for correct answer and (–1/3) marks for wrong answer.

1. Let $f(x)=|x^2-25|$ for all $x\in\mathbb{R}$. The total number of points of $\mathbb{R}$ at which f attains a local extremum (minimum or maximum) is

(a) 1 (b) 2 (c) 3 (d) 4

2. The coefficient of $(x-1)^2$ in the Taylor series expansion of $f(x)=xe^x (x\in\mathbb{R})$ about the point $x=1$ is

(a) $\frac{e}{2}$ (b) $2e$ (c) $\frac{3e}{2}$ (d) $3e$

3. Let $f(x,y)=\sum_{k=1}^{10}(x^2-y^2)^k$ for all $(x,y)\in\mathbb{R}^2$. Then for all $(x,y)\in\mathbb{R}^2$,

(a) $x\frac{\partial f}{\partial x}(x,y)-y\frac{\partial f}{\partial y}(x,y)=0$

(b) $x\frac{\partial f}{\partial x}(x,y)+y\frac{\partial f}{\partial y}(x,y)=0$

(c) $y\frac{\partial f}{\partial x}(x,y)-x\frac{\partial f}{\partial y}(x,y)=0$

(d) $y\frac{\partial f}{\partial x}(x,y)+x\frac{\partial f}{\partial y}(x,y)=0$

4. For $a,b,c\in\mathbb{R}$, if the differential equation $(ax^2+bxy+y^2)dx+(2x^2+cxy+y^2)dy=0$ is exact, then

(a) $b=2,c=2a$ (b) $b=4,c=2$ (c) $b=2,c=4$ (d) $b=2,a=2c$

5. If $f(x,y,z)=x^2y+y^2z+z^2x$ for all $(x,y,z)\in\mathbb{R}^3$ and $\nabla=\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}$, then the value of $\nabla\bullet(\nabla\times\nabla f)+\nabla\bullet(\nabla f)$ at (1, 1, 1) is

(a) 0 (b) 3 (c) 6 (d) 9

6. The radius of convergence of the power series $\sum_{n=0}^{\infty}2^{2n}x^{n^2}$ is

(a) $\frac{1}{4}$ (b) 1 (c) 2 (d) 4

7. Let G be a group of order 17. The total number of non-isomorphic subgroups of G is

(a) 1 (b) 2 (c) 3 (d) 17

8. Which one of the following is a subspace of the vector space $\mathbb{R}^3$?

(a) $\{(x,y,z)\in\mathbb{R}^3 : x+2y=0, 2x+3z=0\}$

(b) $\{(x,y,z)\in\mathbb{R}^3 : 2x+3y+4z-3=0, z=0\}$

(c) $\{(x,y,z)\in\mathbb{R}^3 : x\ge 0, y\ge 0\}$

(d) $\{(x,y,z)\in\mathbb{R}^3 : x-1=0, y=0\}$

9. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(x,y,z)=(x+y,y+z,z+x)$ for all $(x,y,z)\in\mathbb{R}^3$. Then

(a) rank (T) = 0, nullity (T) = 3 (b) rank (T) = 2, nullity (T) = 1

(c) rank (T) = 1, nullity (T) = 2 (d) rank (T) = 3, nullity (T) = 0

10. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying $x+\int_0^x f(t)dt = e^x - 1$ for all $x\in\mathbb{R}$. Then the set $\{x\in\mathbb{R}:1\le f(x)\le 2\}$ is the interval

(a) [log 2, log 3] (b) [2 log 2, 3 log 3] (c) $[e-1, e^2-1]$ (d) $[0, e^2]$

Q.11-Q.35: Only one option is correct. Each question carries (+2) marks for correct answer and (–2/3) marks for wrong answer.

11. The system of linear equations

$$x-y+2z=b_1$$
$$x+2y-z=b_2$$
$$2y-2z=b_3$$

is inconsistent when (b_1,b_2,b_3) equals

(a) (2, 2, 0) (b) (0, 3, 2) (c) (2, 2, 1) (d) (2, –1, –2)

12. Let $A=\begin{bmatrix} a & -1 & 4 \\ 0 & b & 7 \\ 0 & 0 & 3 \end{bmatrix}$ be a matrix with real entries. If the sum and the product of all the eigenvalues of A are 10 and 30 respectively, then a^2+b^2 equals

(a) 29 (b) 40 (c) 58 (d) 65

13. Consider the subspace $W=\{(x_1,x_2,...,x_{10})\in\mathbb{R}^{10}: x_n = x_{n-1}+x_{n-2} \text{ for } 3\le n\le 10\}$ of the vector space $\mathbb{R}^{10}$. The dimension of W is

(a) 2 (b) 3 (c) 9 (d) 10

14. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the differential equation $x^2y''(x)-2xy'(x)-4y(x)=0$ for $x\in[1,10]$.

Consider the Wronskian $W(x)=y_1(x)y_2'(x)-y_2(x)y_1'(x)$. If $W(1)=1$, then $W(3)-W(2)$ equals

(a) 1 (b) 2 (c) 3 (d) 5

15. The equation of the curve passing through the point $\left(\frac{\pi}{2},1\right)$ and having slope $\frac{\sin(x)}{x^2}-\frac{2y}{x}$ at each point (x,y) with $x\neq 0$ is

(a) $-x^2y+\cos(x)=\frac{-\pi^2}{4}$ (b) $x^2y+\cos(x)=\frac{\pi^2}{4}$

(c) $x^2y-\sin(x)=\frac{\pi^2}{4}-1$

(d) $x^2y+\sin(x)=\frac{\pi^2}{4}+1$

16. The value of $\alpha\in\mathbb{R}$ for which the curves $x^2+\alpha y^2=1$ and $y=x^2$ intersect orthogonally is

(a) –2
(b) $\frac{-1}{2}$
(c) $\frac{1}{2}$
(d) 2

17. Let $x_n=2^{2n}\left(1-\cos\left(\frac{1}{2^n}\right)\right)$ for all $n\in\mathbb{N}$. Then the sequence $\{x_n\}$

(a) does NOT converges
(b) converges to 0
(c) converges to $\frac{1}{2}$
(d) converges to $\frac{1}{4}$

18. Let $\{x_n\}$ be a sequence of real numbers such that $\lim_{n\to\infty}(x_{n+1}-x_n)=c$, where c is a positive real number. Then the sequence $\left\{\frac{x_n}{n}\right\}$

(a) is NOT bounded
(b) is bounded but NOT convergent
(c) converges to c
(d) converges to 0

19. Let $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ be two series, where $a_n=\frac{(-1)^n n}{2^n}, b_n=\frac{(-1)^n}{\log(n+1)}$ for all $n\in\mathbb{N}$. Then

(a) both $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ are absolutely convergent

(b) $\sum_{n=1}^{\infty}a_n$ is absolutely convergent but $\sum_{n=1}^{\infty}b_n$ is conditionally convergent

(c) $\sum_{n=1}^{\infty}a_n$ is conditionally convergent but $\sum_{n=1}^{\infty}b_n$ is absolutely convergent

(d) both $\sum_{n=1}^{\infty}a_n$ and $\sum_{n=1}^{\infty}b_n$ are conditionally convergent

20. The set $\left\{\frac{x^2}{1+x^2}:x\in\mathbb{R}\right\}$ is

(a) connected but NOT compact in $\mathbb{R}$
(b) compact but NOT connected in $\mathbb{R}$
(c) compact and connected $\mathbb{R}$
(d) neither compact nor connected in $\mathbb{R}$

21. The set of all limit points of the set $\left\{\frac{2}{x+1}:x\in(-1,1)\right\}$ in $\mathbb{R}$ is

(a) $[1,\infty)$
(b) $(1,\infty)$
(c) $[-1,1]$
(d) $[-1,\infty)$

22. Let $S=[0,1]\cup[2,3)$ and let $f:S\to\mathbb{R}$ be defined by $f(x)=\begin{cases}2x & \text{if } x\in[0,1],\\ 8-2x & \text{if } x\in[2,3).\end{cases}$

If $T=\{f(x):x\in S\}$, then the inverse function $f^{-1}:T\to S$

(a) does NOT exist (b) exists and is continuous

(c) exists and is NOT continuous (d) exists and is monotonic

23. Let $f(x)=x^3+x$ and $g(x)=x^3-x$ for all $x\in\mathbb{R}$. If f^{-1} denotes the inverse function of f, then the derivative of the composite function $g\,o\,f^{-1}$ at the point 2 is

(a) $\frac{2}{13}$ (b) $\frac{1}{2}$ (c) $\frac{11}{13}$ (d) $\frac{11}{4}$

24. For all $(x,y)\in\mathbb{R}^2$, let $f(x,y)=\begin{cases}x & \text{if } y=0,\\ x-y^3\sin(1/y) & \text{if } y\neq 0.\end{cases}$

Then at the point (0, 0),

(a) f is NOT continuous (b) f is continuous but NOT differentiable

(c) $\frac{\partial f}{\partial x}$ exists but $\frac{\partial f}{\partial y}$ does NOT exist (d) f is differentiable

25. For all $(x,y)\in\mathbb{R}^2$, let $f(x,y)=\begin{cases}\frac{x}{|x|}\sqrt{x^2+y^2} & \text{if } x\neq 0\\ 0 & \text{if } x=0.\end{cases}$

Then $\frac{\partial f}{\partial x}(0,0)+\frac{\partial f}{\partial y}(0,0)$ equals

(a) –1 (b) 0 (c) 1 (d) 2

26. Let $f:\mathbb{R}\to\mathbb{R}$ be a function with continuous derivative such that $f(\sqrt{2})=2$ and $f(x)=\lim_{t\to 0}\frac{1}{2t}\int_{x-t}^{x+t} sf'(s)ds$ for all $x\in\mathbb{R}$. Then $f(3)$ equals

(a) $\sqrt{3}$ (b) $3\sqrt{2}$ (c) $3\sqrt{3}$ (d) 9

27. The value of $\int_{x=0}^{1}\int_{y=0}^{x^2}\int_{z=0}^{y}(y+2z)\,dz\,dy\,dx$ is

(a) $\frac{1}{53}$ (b) $\frac{2}{21}$ (c) $\frac{1}{6}$ (d) $\frac{5}{3}$

28. If C is a smooth curve in $\mathbb{R}^3$ from (–1, 0, 1) to (1, 1, –1), then the value of $\int_C (2xy+z^2)dx+(x^2+z)dy+(y+2xz)dz$ is

(a) 0 (b) 1 (c) 2 (d) 3

29. Let C be the boundary of the region $R=\left\{(x,y)\in\mathbb{R}^2:-1\le y\le 1, 0\le x\le 1-y^2\right\}$ oriented in the counterclockwise direction. Then the value of $\oint_C ydx+2xdy$ is

(a) $\frac{-4}{3}$ (b) $\frac{-2}{3}$ (c) $\frac{2}{3}$ (d) $\frac{4}{3}$

30. Let G be a cyclic group of order 24. The total number of group isomorphisms of G onto itself is

(a) 7 (b) 8 (c) 17 (d) 24

31. Let S_n be the group of all permutations on the set $\{1, 2,...,n\}$ under the composition of mappings. For $n>2$, if H is the smallest subgroup of S_n containing the transposition (1, 2) and the cycle (1, 2, ...,n), then

(a) $H=S_n$ (b) H is abelian

(c) the index of H in S_n is 2 (d) H is cyclic

32. Let S be the oriented surface $x^2+y^2+z^2=1$ with the unit normal $\mathbf{n}$ pointing outward. For the vector field $\mathbf{F}(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, the value of $\iint_S \mathbf{F}.\mathbf{n}\,dS$ is

(a) $\frac{\pi}{3}$ (b) 2π (c) $\frac{4\pi}{3}$ (d) 4π

33. Let $f:(0,\infty)\to\mathbb{R}$ be a differentiable function such that $f'(x^2)=1-x^3$ for all $x>0$ and $f(1)=0$. Then $f(4)$ equals

(a) $\frac{-47}{5}$ (b) $\frac{-47}{10}$ (c) $\frac{-16}{5}$ (d) $\frac{-8}{5}$

34. Which one of the following conditions on a group G implies that G is abelian?

(a) The order of G is p^3 for some prime p

(b) Every proper subgroup of G is cyclic

(c) Every subgroup of G is normal in G

(d) The function $f:G\to G$, defined by $f(x)=x^{-1}$ for all $x\in G$, is a homomorphism

35. Let $S=\{x\in\mathbb{R}:x^6-x^5\le 100\}$ and $T=\{x^2-2x:x\in(0,\infty)\}$. The set $S\cap T$ is

(a) closed and bounded in $\mathbb{R}$ (b) closed but NOT bounded in $\mathbb{R}$

(c) bounded but NOT closed in $\mathbb{R}$ (d) neither closed nor bounded in $\mathbb{R}$

PART-II
(DESCRIPTIVE QUESTIONS)

Q.36-Q.43 carry five marks each.

36. Find all the critical points of the function $f:\mathbb{R}^2 \to \mathbb{R}$ defined by $f(x,y) = x^3 + xy + y^3$ for all $(x,y) \in \mathbb{R}^2$. Also, examine whether the function f attains a local maximum or a local minimum at each of these critical points.

37. Given that there is a common solution to the following equations:

$\mathbf{P}: y' + 2y = e^x y^2, y(0) = 1,$

$\mathbf{Q}: y'' - 2y' + \alpha y = 0,$

find the value of α and hence find the general solution of **Q**.

38. Let $f:\mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $f\left(\frac{1}{2^n}\right) = 0$ for all $n \in \mathbb{N}$. Show that $f'(0) = 0 = f''(0)$.

39. Let A be an $n \times n$ matrix with real entries such that $A^2 = A$. If I denotes the $n \times n$ identity matrix, then show that rank($A - I$) = nullity (A).

40. Evaluate $\iint_S \frac{xy}{\sqrt{1+2x^2}} dS$, where the surface $S = \{(x, y, x^2 + y) \in \mathbb{R}^3 : 0 \le x \le y, x + y \le 1\}$.

41. Let $f:(0,1) \to \mathbb{R}$ be a differentiable function such that $|f'(x)| \le 5$ for all $x \in (0,1)$. Show that the sequence $\left\{f\left(\frac{1}{n+1}\right)\right\}$ converges in $\mathbb{R}$.

42. Let H be a subgroup of the group ($\mathbb{R}$, +) such that $H \cap [-1,1]$ is a finite set containing a non-zero element. Show that H is cyclic.

43. If K is a nonempty closed subset of $\mathbb{R}$, then show that the set $\{x + y : x \in K, y \in [1,2]\}$ is closed in $\mathbb{R}$.

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