Section-A

Multiple Choice Questions (MCQ)

Q.1 – Q.10 carry ONE mark each.

1. The sequence $\{s_n\}$ of real numbers given by

$$s_n = \frac{\sin\frac{\pi}{2}}{1\cdot 2} + \frac{\sin\frac{\pi}{2^2}}{2\cdot 3} + \cdots + \frac{\sin\frac{\pi}{2^n}}{n\cdot(n+1)} \text{ is}$$

(a) a divergent sequence (b) an oscillatory sequence
(c) not a Cauchy sequence (d) a Cauchy sequence

2. Let P be the vector space (over $\mathbb{R}$) of all polynomials of degree ≤ 3 with real coefficients. Consider the linear transformation $T : P \to P$ defined by $T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_3 + a_2x + a_1x^2 + a_0x^3$. Then the matrix representation M of T with respect to the ordered basis $\{1, x, x^2, x^3\}$ satisfies

(a) $M^2 + I_4 = 0$ (b) $M^2 - I_4 = 0$ (c) $M - I_4 = 0$ (d) $M + I_4 = 0$

3. Let $f : [-1, 1] \to \mathbb{R}$ be a continuous function. Then the integral $\int_0^\pi x\, f(\sin x)\, dx$ is equivalent to

(a) $\frac{\pi}{2}\int_0^\pi f(\sin x)\, dx$ (b) $\frac{\pi}{2}\int_0^\pi f(\cos x)\, dx$ (c) $\pi\int_0^\pi f(\cos x)\, dx$ (d) $\pi\int_0^\pi f(\sin x)\, dx$

4. Let σ be an element of the permutation group S_5. Then the maximum possible order of σ is
(a) 5 (b) 6 (c) 10 (d) 15

5. Let f be a strictly monotonic continuous real valued function defined on $[a, b]$ such that $f(a) < a$ and $f(b) > b$. Then which one of the following is TRUE ?

(a) There exists exactly one $c \in (a, b)$ such that $f(c) = c$
(b) There exists exactly two points $c_1, c_2 \in (a, b)$ such that $f(c_i) = c_i, i = 1, 2$
(c) There exists no $c \in (a, b)$ such that $f(c) = c$
(d) There exist infinitely many points $c \in (a, b)$ such that $f(c) = c$

6. The value of $\lim\limits_{(x,y)\to(2,-2)} \frac{\sqrt{(x-y)}-2}{x-y-4}$ is

(a) 0 (b) $\frac{1}{4}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

7. Let $\vec{r} = (x\hat{i} + y\hat{j} + z\hat{k})$ and $r = |\vec{r}|$. If $f(r) = \ell n\, r$ and $g(r) = \frac{1}{r}, r \ne 0$ satisfy $2\,\nabla f + h(r)\,\nabla g = \vec{0}$, then $h(r)$ is

(a) r (b) $\frac{1}{r}$ (c) $2r$ (d) $\frac{2}{r}$

8. The non-zero value of n for which the differential equation $(3xy^2 + n^2x^2y)\,dx + (nx^3 + 3x^2y)\,dy = 0,\ x \ne 0$, becomes exact is
(a) –3 (b) –2 (c) 2 (d) 3

9. One of the points which lies on the solution curve of the differential equation $(y-x)\,dx+(x+y)\,dy=0$, with the given condition $y(0)=1$, is

(a) (1, –2) (b) (2, –1) (c) (2, 1) (d) (–1, 2)

10. Let S be a closed subset of $\mathbb{R}$, T a compact subset of $\mathbb{R}$ such that $S \cap T \neq \phi$. Then $S \cap T$ is

(a) closed but not compact (b) not closed
(c) compact (d) neither closed nor compact

Q.11 – Q.30 carry TWO marks each.

11. Let S be the series $\sum_{k=1}^{\infty} \frac{1}{(2k-1)\,2^{(2k-1)}}$ and T be the series $\sum_{k=2}^{\infty} \left(\frac{3k-4}{3k+2}\right)^{\frac{(k+1)}{3}}$ of real numbers. Then which one of the following is TRUE ?

(a) Both the series S and T are convergent (b) S is convergent and T is divergent
(c) S is divergent and T is convergent (d) Both the series S and T are divergent

12. Let $\{a_n\}$ be a sequence of positive real numbers satisfying $\frac{4}{a_{n+1}} = \frac{3}{a_n} + \frac{a_n^3}{81}$, $n \geq 1$, $a_1 = 1$. Then all the terms of the sequence lie in

(a) $\left[\frac{1}{2}, \frac{3}{2}\right]$ (b) $[0, 1]$ (c) $[1, 2]$ (d) $[1, 3]$

13. The largest eigenvalue of the matrix $\begin{bmatrix} 1 & 4 & 16 \\ 4 & 16 & 1 \\ 16 & 1 & 4 \end{bmatrix}$ is

(a) 16 (b) 21 (c) 48 (d) 64

14. The value of the integral $\frac{(2n)!}{2^{2n}(n!)} \int_{-1}^{1} (1-x^2)^n\, dx,\ n \in \mathbb{N}$ is

(a) $\frac{2}{(2n+1)!}$ (b) $\frac{2n}{(2n+1)!}$ (c) $\frac{2(n!)}{2n+1}$ (d) $\frac{(n+1)!}{2n+1}$

15. If the triple integral over the region bounded by the planes $2x+y+z=4$, $x=0$, $y=0$, $z=0$ is given by $\int_0^2 \int_0^{\lambda(x)} \int_0^{\mu(x,y)} dz\,dy\,dx$, then the function $\lambda(x)-\mu(x,y)$ is

(a) $x+y$ (b) $x-y$ (c) x (d) y

16. The surfaces area of the portion of the plane $y+2z=2$ within the cylinder $x^2+y^2=3$ is

(a) $\frac{3\sqrt{5}}{2}\pi$ (b) $\frac{5\sqrt{5}}{2}\pi$ (c) $\frac{7\sqrt{5}}{2}\pi$ (d) $\frac{9\sqrt{5}}{2}\pi$

17. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be defined by $f(x,y) = \begin{cases} \frac{xy^2}{x+y} & \text{if } x+y \neq 0 \\ 0 & \text{if } x+y=0 \end{cases}$. Then the value of $\left(\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y \partial x}\right)$ at the point (0, 0) is

(a) 0 (b) 1 (c) 2 (d) 4

18. The function $f(x, y) = 3x^2y + 4y^3 - 3x^2 - 12y^2 + 1$ has a saddle point at
(a) $(0, 0)$ (b) $(0, 2)$ (c) $(1, 1)$ (d) $(-2, 1)$

19. Consider the vector field $\vec{F} = r^{\beta}(y\,\hat{i} - x\,\hat{j})$, where $\beta \in \mathbb{R}$, $\vec{r} = x\,\hat{i} + y\,\hat{j}$ and $r = |\vec{r}|$. If the absolute value of the line integral $\oint_C \vec{F} \cdot d\vec{r}$ along the closed curve $C : x^2 + y^2 = a^2$ (oriented counter clockwise) is 2π, then β is
(a) -2 (b) -1 (c) 1 (d) 2

20. Let S be the surface of the cone $z = \sqrt{x^2 + y^2}$ bounded by the planes $z = 0$ and $z = 3$. Further, let C be the closed curve forming the boundary of the surfaces S. A vector field $\vec{F}$ is such that $\nabla \times \vec{F} = -x\,\hat{i} - y\,\hat{j}$. The absolute value of the integral $\oint_C \vec{F} \cdot d\vec{r}$, where $\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k}$ and $r = |\vec{r}|$, is
(a) 0 (b) 9π (c) 15π (d) 18π

21. Let $y(x)$ be the solution of the differential equation $\frac{d}{dx}\left(x\frac{dy}{dx}\right) = x$; $y(1) = 0$, $\left.\frac{dy}{dx}\right|_{x=1} = 0$. Then $y(2)$ is
(a) $\frac{3}{4} + \frac{1}{2}\ell n\, 2$ (b) $\frac{3}{4} - \frac{1}{2}\ell n\, 2$ (c) $\frac{3}{4} + \ell n\, 2$ (d) $\frac{3}{4} - \ell n\, 2$

22. The general solution of the differential equation with constant co-efficients $\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$, approaches zero as $x \to \infty$, if
(a) b is negative and c is positive
(b) b is positive and c is negative
(c) both b and c are positive
(d) both b and c are negative

23. Let $S \subset \mathbb{R}$ and ∂S denote the set of points x in $\mathbb{R}$ such that every neighbourhood of x contains some points of S as well as some points of complement of S. Further, let $\overline{S}$ denote the closure of S. Then which one of the following is FALSE ?
(a) $\partial\mathbb{Q} = \mathbb{R}$
(b) $\partial(\mathbb{R}\backslash T) = \partial T,\ T \subset \mathbb{R}$
(c) $\partial(T \cup V) = \partial T \cup \partial V,\ T, V \subset \mathbb{R},\ T \cap V \neq \phi$
(d) $\partial T = \overline{T} \cap (\overline{\mathbb{R}\backslash T}),\ T \subset \mathbb{R}$

24. The sum of the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 + n - 2}$ is
(a) $\frac{1}{3}\ell n\, 2 - \frac{5}{18}$ (b) $\frac{1}{3}\ell n\, 2 - \frac{5}{6}$ (c) $\frac{2}{3}\ell n\, 2 - \frac{5}{18}$ (d) $\frac{2}{3}\ell n\, 2 - \frac{5}{6}$

25. Let $f(x) = \frac{1}{1+|x|} + \frac{1}{1+|x-1|}$ for all $x \in [-1, 1]$. Then which one of the following is TRUE ?
(a) Maximum value of $f(x)$ is $\frac{3}{2}$
(b) Minimum value of $f(x)$ is $\frac{1}{3}$
(c) Maximum of $f(x)$ occurs at $x = \frac{1}{2}$
(d) Minimum of $f(x)$ occurs at $x = 1$

26. The matrix $M = \begin{bmatrix} \cos\alpha & \sin\alpha \\ i\sin\alpha & i\cos\alpha \end{bmatrix}$ is a unitary matrix when α is

(a) $(2n+1)\frac{\pi}{2}, n \in \mathbb{Z}$ (b) $(3n+1)\frac{\pi}{3}, n \in \mathbb{Z}$ (c) $(4n+1)\frac{\pi}{4}, n \in \mathbb{Z}$ (d) $(5n+1)\frac{\pi}{5}, n \in \mathbb{Z}$

27. Let $M = \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & \alpha \\ 2 & -\alpha & 0 \end{bmatrix}$, $\alpha \in \mathbb{R}\backslash\{0\}$ and b a non-zero vector such that $Mx = b$ for some $x \in \mathbb{R}^3$. Then the value of $x^T b$ is

(a) $-\alpha$ (b) α (c) 0 (d) 1

28. The number of group homomorphism from the cyclic group $\mathbb{Z}_4$ to the cyclic group $\mathbb{Z}_7$ is

(a) 7 (b) 3 (c) 2 (d) 1

29. In the permutation group S_n $(n \ge 5)$, if H is the smallest subgroup containing all the 3-cycles, then which one of the following is TRUE ?

(a) Order of H is 2 (b) Index of H in S_n is 2

(c) H is abelian (d) $H = S_n$

30. Let $f:\mathbb{R} \to \mathbb{R}$ be defined as $f(x) = \begin{cases} x(1 + x^{\alpha}\sin(\ell n\, x^2)) & \text{if} \quad x \ne 0 \\ 0 & \text{if} \quad x = 0 \end{cases}$. Then, at $x = 0$, the function f is

(a) continuous and differentiable when $\alpha = 0$

(b) continuous and differentiable when $\alpha > 0$

(c) continuous and differentiable when $-1 < \alpha < 0$

(d) continuous and differentiable when $\alpha < -1$

Section-B

Multiple Select Questions (MSQ)

Q.31 – Q.40 carry TWO marks each.

31. Let $\{s_n\}$ be a sequence of positive real numbers satisfying $2\,s_{n+1} = s_n^2 + \frac{3}{4},\ n \ge 1$. If α and β are the roots of the equation $x^2 - 2x + \frac{3}{4} = 0$ and $\alpha < s_1 < \beta$, then which of the following statement(s) is/are TRUE ?

(a) $\{s_n\}$ is monotonically decreasing (b) $\{s_n\}$ is monotonically increasing

(c) $\lim_{n\to\infty} s_n = \alpha$ (d) $\lim_{n\to\infty} s_n = \beta$

32. The value(s) of the integral $\int_{-\pi}^{\pi} |x| \cos nx\, dx,\ n \ge 1$ is(are)

(a) 0 when n is even (b) 0 when n is odd

(c) $-\frac{4}{n^2}$ when n is even (d) $-\frac{4}{n^2}$ when n is odd

33. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y)=\begin{cases}\dfrac{xy}{|x|} & \text{if } x\neq 0\\ 0 & \text{elsewhere}\end{cases}$$

Then at the point (0, 0), which of the following statement(s) is(are) TRUE ?

(a) f is not continuous
(b) f is continuous
(c) f is differentiable
(d) Both first order partial derivatives of f exist

34. Consider the vector field $\vec{F}=x\hat{i}+y\hat{j}$ on an open connected set $S\subset\mathbb{R}^2$. Then which of the following statement(s) is(are) TRUE ?

(a) Divergence of $\vec{F}$ is zero on S
(b) The line integral of $\vec{F}$ is independent of path in S
(c) $\vec{F}$ can be expressed as a gradient of a scalar function on S
(d) The line integral of $\vec{F}$ is zero around any piecewise smooth closed path in S

35. Consider the differential equation $\sin 2x\dfrac{dy}{dx}=2y+2\cos x,\ y\left(\dfrac{\pi}{4}\right)=1-\sqrt{2}$. Then which of the following statement(s) is(are) TRUE ?

(a) The solution is unbounded when $x\to 0$
(b) The solution is unbounded when $x\to\dfrac{\pi}{2}$
(c) The solution is bounded when $x\to 0$
(d) The solution is bounded when $x\to\dfrac{\pi}{2}$

36. Which of the following statement(s) is(are) TRUE ?

(a) There exists a connected set in $\mathbb{R}$ which is not compact
(b) Arbitrary union of closed intervals in $\mathbb{R}$ need not be compact
(c) Arbitrary union of closed intervals in $\mathbb{R}$ is always closed
(d) Every bounded infinite subset V of $\mathbb{R}$ has a limit point in V itself

37. Let $P(x)=\left(\dfrac{5}{13}\right)^x+\left(\dfrac{12}{13}\right)^x-1$ for all $x\in\mathbb{R}$. Then which of the following statement(s) is(are) TRUE ?

(a) The equation $P(x)=0$ has exactly one solution in $\mathbb{R}$
(b) $P(x)$ is strictly increasing for all $x\in\mathbb{R}$
(c) The equation $P(x)=0$ has exactly two solutions in $\mathbb{R}$
(d) $P(x)$ is strictly decreasing for all $x\in\mathbb{R}$

38. Let G be a finite group and O(G) denotes its order. Then which of the following statement(s) is(are) TRUE ?

(a) G is abelian if O(G) denotes = pq where p and q are distinct primes
(b) G is abelian if every non identity element of G is of order 2
(c) G is abelian if the quotient group G/Z(O) is cyclic, where Z(G) is the center of G
(d) G is abelian if O(G) = p^3, where p is prime

39. Consider the set $V=\left\{\begin{bmatrix}x\\y\\z\end{bmatrix}\in\mathbb{R}^3 \mid \alpha x+\beta y+z=\gamma,\ \alpha,\beta,\gamma\in\mathbb{R}\right\}$. For which of the following choice(s) the set V becomes a two dimensional subspace of $\mathbb{R}^3$ over $\mathbb{R}$?

(a) $\alpha=0,\ \beta=1,\ \gamma=0$
(b) $\alpha=0,\ \beta=1,\ \gamma=1$
(c) $\alpha=1,\ \beta=0,\ \gamma=0$
(d) $\alpha=1,\ \beta=1,\ \gamma=0$

40. Let $S = \left\{ \frac{1}{3^n} + \frac{1}{7^m} \middle| n, m \in \mathbb{N} \right\}$. Then which of the following statement(s) is(are) TRUE ?

(a) S is closed (b) S is not open (c) S is connected (d) 0 is a limit point of S

Section-C

Numerical Answer Type (NAT)

Q.41 – Q.50 carry ONE mark each.

41. Let $\{s_n\}$ be a sequence of real numbers given by $s_n = 2^{(-1)^n}\left(1 - \frac{1}{n}\right)\sin\frac{n\pi}{2}$, $n \in \mathbb{N}$. Then the least upper bound of the sequence $\{s_n\}$ is_______.

42. Let $\{s_k\}$ be a sequence of real numbers, where $s_k = k^{\alpha/k}$, $k \geq 1$, $\alpha = 0$. Then $\lim_{n\to\infty} (s_1\, s_2 \ldots s_n)^{1/n}$ is_______.

43. Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ be a non-zero vector and $A = \frac{xx^T}{x^T x}$. Then the dimension of the vector space $\left\{ y \in \mathbb{R}^3 \mid Ay = 0 \right\}$ over $\mathbb{R}$ is_______.

44. Let f be a real valued function defined by $f(x, y) = 2\,\ell n\left(x^2 y^2 e^{y/x}\right)$, $x > 0$, $y > 0$, is_______.

45. Let $\vec{F} = \sqrt{x}\,\hat{i} + (x + y^3)\,\hat{j}$ be a vector field for all (x, y) with $x \geq 0$ and $\vec{r} = x\,\hat{i} + y\,\hat{j}$. Then the value of the line integral $\int_C \vec{F} \cdot d\vec{r}$ from $(0, 0)$ to $(1, 1)$ along the path $C: x = t^2,\ y = t^3,\ 0 \leq t \leq 1$ is_______.

46. If $f: (-1, \infty) \to \mathbb{R}$ defined by $f(x) = \frac{x}{1+x}$ is expressed as $f(x) = \frac{2}{3} + \frac{1}{9}(x-2) + \frac{c\,(x-2)^2}{(1+\xi)^3}$, where ξ lies between 2 and x, then the value of c is_______.

47. Let $y_1(x)$, $y_2(x)$ and $y_3(x)$ be linearly independent solutions of the differential equation

$$\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0.$$

If the Wronskian $W(y_1, y_2, y_3)$ is of the form $k\,e^{bx}$ for some constant k, then the value of b is_______.

48. The radius of convergence of the power series $\sum_{n=1}^{\infty} \frac{(-4)^n}{n(n+1)}(x+2)^{2n}$ is_______.

49. Let $f: (0, \infty) \to \mathbb{R}$ be a continuous function such that $\int_0^x f(t)\,dt = -2 + \frac{x^2}{2} + 4x\sin 2x + 2\cos 2x.$ Then the value of $\frac{1}{\pi} f\left(\frac{\pi}{4}\right)$ is_______.

50. Let G be a cyclic group of order 12. Then the number of non-isomorphic subgroups of G is_______.

Q.51 – Q.60 carry TWO marks each.

51. The value of $\lim_{n\to\infty}\left(8n-\frac{1}{n}\right)^{\frac{(-1)^n}{n^2}}$ is equal to_______.

52. Let R be the region enclosed by $x^2+4y^2\geq 1$ and $x^2+y^2\leq 1$. Then the value of $\iint_R |xy|\,dx\,dy$ is_______.

53. Let $M=\begin{bmatrix}\alpha & 1 & 1\\ 1 & \beta & 1\\ 1 & 1 & \gamma\end{bmatrix}$, $\alpha\beta\gamma=1$, $\alpha,\beta,\gamma\in\mathbb{R}$ and $x=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}\in\mathbb{R}^3$. Then $Mx=0$ has infinitely many solutions if trace (M) is_______.

54. Let C be the boundary of the region enclosed by $y=x^2$, $y=x+2$ and $x=0$. Then the value of the line integral $\oint_C (xy-y^2)\,dx - x^3dy$, where C is traversed in the counter clockwise direction, is_______.

55. Let S be the closed surface forming the boundary of the region V bounded by $x^2+y^2=3$, $z=0$, $z=6$. A vector field $\vec{F}$ is defined over V with $\nabla\cdot\vec{F}=2y+z+1$. Then the value of $\frac{1}{\pi}\iint_S \vec{F}\cdot\hat{n}\,dS$, where $\hat{n}$ is the unit outward drawn normal to the surface S, is_______.

56. Let $y(x)$ be the solution of the differential equation $\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0$, $y(0)=1$, $\left.\frac{dy}{dx}\right|_{x=0}=-1$. Then $y(x)$ attains its maximum value at $x=$_______.

57. The value of the double integral $\int_0^{\pi}\int_0^{x}\frac{\sin y}{\pi-y}\,dy\,dx$ is_______.

58. Let H denote the group of all 2×2 invertible matrices over $\mathbb{Z}_5$ under usual matrix multiplication. Then the order of the matrix $\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$ in H is_______.

59. Let $A=\begin{bmatrix}1 & 2 & 0\\ -1 & 5 & 2\end{bmatrix}$, $B=\begin{bmatrix}1 & 2\\ -1 & 0\\ 3 & 1\end{bmatrix}$, $N(A)$ the null space of A and $R(B)$ the range space of B. Then the dimension of $N(A)\cap R(B)$ over $\mathbb{R}$ is_______.

60. The maximum value of $f(x,y)=x^2+2y^2$ subject to the constraint $y-x^2+1=0$ is_______.

*** **END OF THE QUESTION PAPER** ***