PAPER : IIT-JAM 2005

MATHEMATICS-MA

(CODE-A)

Q.1-Q.15: Only one option is correct for each question. Each question carries (+6) marks for correct answer and (–2) marks for incorrect answer.

1. Let $\{a_n\}, \{b_n\}$ and $\{c_n\}$ be sequences of real numbers such that $b_n = a_{2n}$ and $c_n = a_{2n+1}$. Then $\{a_n\}$ is convergent

(a) implies $\{b_n\}$ is convergent but $\{c_n\}$ need not be convergent

(b) implies $\{c_n\}$ is convergent but $\{b_n\}$ need not be convergent

(c) implies both $\{b_n\}$ and $\{c_n\}$ are convergent

(d) if both $\{b_n\}$ and $\{c_n\}$ are convergent

2. An integrating factor of $x\dfrac{dy}{dx} + (3x+1)y = xe^{-2x}$ is

(a) xe^{3x} (b) $3xe^{x}$ (c) xe^{x} (d) x^3e^{x}

3. The general solution of $x^2\dfrac{d^2y}{dx^2} - 5x\dfrac{dy}{dx} + 9y = 0$ is

(a) $(c_1 + c_2 x)e^{3x}$ (b) $(c_1 + c_2 \ln x)x^3$ (c) $(c_1 + c_2 x)x^3$ (d) $(c_1 + c_2 \ln x)e^{x^3}$

4. Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. If $\phi(x, y, z)$ is a solution of the Laplace equation then the vector field $(\vec{\nabla}\phi + \vec{r})$ is

(a) neither solenoidal nor irrotational (b) solenoidal but not irrotational
(c) both solenoidal and irrotational (d) irrotational but not solenoidal

5. Let $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, S be the surface of the sphere $x^2 + y^2 + z^2 = 1$ and $\hat{n}$ be the inward unit normal vector to S. Then $\iint_S \vec{F}.\hat{n}dS$ is equal to

(a) 4π (b) -4π (c) 8π (d) -8π

6. Let A be a 3 × 3 matrix with eigen values 1, –1 and 3. Then

(a) $A^2 + A$ is non-singular (b) $A^2 - A$ is non-singular

(c) $A^2 + 3A$ is non-singular (d) $A^2 - 3A$ is non-singular

7. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation and I be the identify transformation of $\mathbb{R}^3$. If there is a scalar C and a non-zero vector $x \in \mathbb{R}^3$ such that $T(x) = Cx$, then rank $(T - CI)$

(a) cannot be 0 (b) cannot be 1 (c) cannot be 2 (d) cannot be 3

8. In the group {1, 2, ...16} under the operation of multiplication modulo 17, the order of the element 3 is

(a) 4 (b) 8 (c) 12 (d) 16

9. A ring R has maximal ideals

(a) if R is infinite (b) if R is finite

(c) if R is finite with at least 2 elements (d) only if R is finite

10. The integral $\int_0^1 \left[\int_0^{1-z}\left(\int_0^2 dx\right)dy\right]dz$ is equal to

(a) $\int_0^1 \left[\int_0^{1-y}\left(\int_0^2 dx\right)dz\right]dy$ (b) $\int_0^1 \left[\int_0^{1-y}\left(\int_0^1 dx\right)dz\right]dy$

(c) $\int_0^2 \left[\int_0^2\left(\int_0^{1-z} dx\right)dz\right]dy$ (d) $\int_0^2 \left[\int_0^2\left(\int_0^{1-y} dx\right)dz\right]dy$

11. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous and $g,h:\mathbb{R}^2\to\mathbb{R}$ be differentiable. Let $F(u,v)=\int_v^u f(t)\,dt$, where $u=g(x,y)$ and $v=h(x,y)$. Then $\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}=$

(a) $f\left(g(x,y)\right)\left[\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}\right]-f\left(h(x,y)\right)\left[\frac{\partial h}{\partial x}+\frac{\partial h}{\partial y}\right]$

(b) $f\left(h(x,y)\right)\left[\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}\right]-f\left(g(x,y)\right)\left[\frac{\partial h}{\partial x}-\frac{\partial h}{\partial y}\right]$

(c) $f\left(h(x,y)\right)\left[\frac{\partial g}{\partial x}+\frac{\partial g}{\partial y}\right]+f\left(g(x,y)\right)\left[\frac{\partial h}{\partial x}-\frac{\partial h}{\partial y}\right]$

(d) $f\left(g(x,y)\right)\left[\frac{\partial g}{\partial x}-\frac{\partial g}{\partial y}\right]+f\left(g(x,y)\right)\left[\frac{\partial h}{\partial x}-\frac{\partial h}{\partial y}\right]$

12. Let $y=f(x)$ be a smooth where such that $0<f(x)<K$ for all $x\in[a,b]$. Let

L = length of the curve between $x = a$ and $x = b$

A = area bounded by the curve, x-axis and the lines $x = a$ and $x = b$

S = area of the surface generates by revolving the curve about x-axis between $x = a$ and $x = b$

Then

(a) $2\pi KL<S<2\pi A$ (b) $S\le 2\pi A<2\pi KL$

(c) $2\pi A\le S<2\pi KL$ (d) $2\pi A<2\pi KL<S$

13. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by $f(t)=t^2$ and let U be any non-empty open subset of $\mathbb{R}$. Then

(a) $f(U)$ is open (b) $f^{-1}(U)$ is open (c) $f(U)$ is closed (d) $f^{-1}(U)$ is closed

14. Let $f:(-1,1)\to\mathbb{R}$ be such that $f^{(n)}(x)$ exists and $\left|f^{(n)}(x)\right|\le 1$ for every $n\ge 1$ and for every $x\in(-1,1)$. Then f has a convergent power series expansion in a neighbourhood of

(a) every $x\in(-1,1)$ (b) every $x\in\left(\frac{-1}{2},0\right)$ only

(c) no $x\in(-1,1)$ (d) every $x\in\left(0,\frac{1}{2}\right)$ only

15. Let $a>1$ and and $f,g,h:[-a,a]\to\mathbb{R}$ be twice differentiable functions such that for some 'c' with $0<c<1<a$,

$$f(x)=0 \text{ only for } x=-a,0,a;$$
$$f'(x)=0=g(x) \text{ only } x=-1,0,1;$$
$$g'(x)=0=h(x) \text{ only for } x=-c,c.$$

The possible relations between f,g,h are

(a) $f=g'$ and $h=f'$ (b) $f'=g$ and $g'=h$

(c) $f=-g'$ and $h'=g$ (d) $f=-g'$ and $h'=f$

16. **(a)** Solve the initial value problem $\dfrac{d^2y}{dx^2}-y=x(\sin x+e^x)$, $y(0)=y'(0)=1$ (12)

(b) Solve the differential equation $(2y\sin x+3y^4\sin x\cos x)dx-(4y^3\cos^2 x+\cos x)dy=0$ (9)

17. Let G be a finite abelian group of order n with identity e. If for all $a\in G, a^3=e$, then, by induction on n, show that $n=3^k$ for some nonnegative integer k. (21)

18. **(a)** Let $f:[a,b]\to\mathbb{R}$ be a differentiable function. Show that there exist two points $c_1,c_2\in(a,b)$ such that $2f(c_1)\,f'(c_1)=f'(c_2)\left[f(a)+f(b)\right]$. (9)

(b) Let $f(x,y)=\begin{cases}\left(x^2+y^2\right)\left[\ell n\left(x^2+y^2\right)+1\right] & \text{for } (x,y)\neq(0,0)\\ \alpha & \text{for } (x,y)=(0,0)\end{cases}$

Find a suitable value of α such that f is continuous. For this value of α, is f differentiable at (0, 0) ? Justify your claim. (12)

19. **(a)** Let S be the surface $x^2+y^2+z^2=1, z\ge 0$. Use Stoke's theorem to evaluate

$$\int_C\left[(2x-y)\,dx-ydy-zdz\right]$$

where C is the circle $x^2+y^2=1, z=0$, oriented anticlockwise. (12)

(b) Show that the vector $\vec{F}=\left(2xy-y^4+3\right)\hat{i}+\left(x^2-4xy^3\right)\hat{j}$ is conservative. Find its potential and also the work done in moving a particle from (1, 0) to (2, 1) along some curve. (9)

20. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ is defined by $T(x,y,z)=(y+z,z,0)$. Show that T is a linear transformation. If $v \in \mathbb{R}^3$ is such that $T^2(v) \neq 0$, then show that $B=\{v,T(v),T^2(v)\}$ forms a basis of $\mathbb{R}^3$. Compute the matrix of T with respect to B. Also find a $v \in \mathbb{R}^3$ such that $T^2(v) \neq 0$. (21)

21. **(a)** For each $n \in N$, define $f_n:[-1,1] \to \mathbf{R}$ by

$$f_n(x)=\begin{cases} 4n^2x & \text{for } x \in \left[0,\dfrac{1}{2n}\right) \\ -4n^2\left(x-\dfrac{1}{n}\right) & \text{for } x \in \left[\dfrac{1}{2n},\dfrac{1}{n}\right) \\ 0 & \text{for } x \in \left[\dfrac{1}{n},1\right] \end{cases}$$

Compute $\int_0^1 f_n(x)\,dx$ for each n. Analyze pointwise and uniform convergence of the sequence of functions $\{f_n\}$. (12)

(b) Let $f:\mathbb{R} \to \mathbb{R}$ be a continuous function with $|f(x)-f(y)| \geq |x-y|$ for every $x, y \in \mathbb{R}$. Is f one-one? Show that there cannot exist three points $a,b,c \in \mathbb{R}$ with $a<b<c$ such that $f(a)<f(c)<f(b)$. (9)

22. Find the volume of the cylinder with base as the disk of unit radius in the *xy*-plane centred at (1, 1, 0) and the top being the surface. $z=[(x-1)^2+(y-1)^2]^{3/2}$ (12)