PAPER : IIT-JAM 2007

MATHEMATICS-MA

(CODE–B)

Q.1-Q.15: Only one option is correct for each question. Each question carries (+6) marks for correct answer and (–2) marks for incorrect answer.

1. Let A(t) denote the area bounded by the curve $y = e^{-|x|}$, the x-axis and the straight lines $x = -t$ and $x = t$, then $\lim_{t\to\infty} A(t)$ is equal to

(a) 2 (b) 1 (c) $\frac{1}{2}$ (d) 0

2. If k is constant such that $xy + k = e^{(x-1)^2/2}$ satisfies the differential equation $x\frac{dy}{dx} = (x^2 - x - 1)y + (x-1)$, then k is equal to

(a) 1 (b) 0 (c) –1 (d) –2

3. Which of the following functions is uniformly continuous on the domain as stated ?

(a) $f(x) = x^2,\ x \in \mathbb{R}$ (b) $f(x) = \frac{1}{x},\ x \in [1, \infty)$

(c) $f(x) = \tan x,\ x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ (d) $f(x) = [x],\ x \in [0, 1]$

([x] is the greatest integer less than or equal to x)

4. Let R be the ring of polynomials over $\mathbb{Z}_2$ and let I be the ideal of R generated by the polynomial $x^3 + x + 1$. Then the number of elements in the quotient ring R/I is

(a) 2 (b) 4 (c) 8 (d) 16

5. Which of the following sets is the basis for the subspace

$W = \left\{ \begin{bmatrix} x & y \\ 0 & t \end{bmatrix} ; x + 2y + t = 0, y + t = 0 \right\}$ of the vector space of all real 2 × 2 matrices.

(a) $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ (b) $\left\{ \begin{bmatrix} 2 & 1 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right\}$

(c) $\left\{ \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \right\}$ (d) $\left\{ \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \right\}$

6. Let G be an Abelian group of order 10. Let $S = \{ g \in G : g^{-1} = g \}$. Then the number of non-identity elements in S is

(a) 5 (b) 2 (c) 1 (d) 0

7. Let (a_n) be an increasing sequence of positive real numbers such that the series $\sum_{k=1}^{\infty} a_k$ is divergent.

Let $S_n = \sum_{k=1}^{n} a_k$ for n = 1, 2, ... and $t_n = \sum_{k=2}^{n} \frac{a_k}{S_{k-1} S_k}$ for n = 2, 3, ... Then $\lim_{n\to\infty} t_n$ is equal to

(a) $\frac{1}{a_1}$ (b) 0 (c) $\frac{1}{(a_1 + a_2)}$ (d) $a_1 + a_2$

8. For every function $f:[0,1]\to\mathbb{R}$ which is twice differentiable and satisfies $f'(x)\geq 1$ for all $x\in[0,1]$, we must have

(a) $f''(x)\geq 0$ for all $x\in[0,1]$

(b) $f(x)\geq x$ for all $x\in[0,1]$

(c) $f(x_2)-x_2 \leq f(x_1)-x_1$ for all $x_1, x_2 \in[0,1]$ with $x_2 \geq x_1$

(d) $f(x_2)-x_2 \geq f(x_1)-x_1$ for all $x_1, x_2 \in[0,1]$ with $x_2 \geq x_1$

9. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0), \\ 0 & \text{if } (x, y) = (0, 0). \end{cases}$$

Which of the following statements holds regarding the continuity and the existence of partial derivatives of *f* at (0, 0) ?

(a) Both partial derivatives of *f* exist at (0, 0) and *f* is continuous at (0, 0)

(b) Both partial derivatives of *f* exist at (0, 0) and *f* is NOT continuous at (0, 0)

(c) One partial derivative of *f* does NOT exist at (0, 0) and *f* is continuous at (0, 0)

(d) One partial derivative of *f* does NOT exist at (0, 0) and *f* is NOT continuous at (0, 0)

10. Suppose (c_n) is a sequence of real numbers such that $\lim_{n\to\infty} |c_n|^{1/n}$ exists and is non-zero. If the radius of convergence of the power series $\sum_{n=0}^{\infty} c_n x^n$ is equal to *r*, then the radius of convergence of the power series $\sum_{n=1}^{\infty} n^2 c_n x^n$ is

(a) less than *r* (b) greater than *r* (c) equal to *r* (d) equal to 0

11. The rank of the matrix $\begin{bmatrix} 1 & 4 & 8 \\ 2 & 10 & 22 \\ 0 & 4 & 12 \end{bmatrix}$ is

(a) 3 (b) 2 (c) 1 (d) 0

12. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function. If $\int_0^x f(2t)dt = \frac{x}{\pi}\sin(\pi x)$ for all $x\in\mathbb{R}$, then $f'(2)$ is equal to

(a) –1 (b) 0 (c) 1 (d) 2

13. Let $\vec{u} = \left(ae^x \sin y - 4x\right)\hat{i} + \left(2y + e^x \cos y\right)\hat{j} + az\hat{k}$, where a is constant. If the line integral $\oint_C \vec{u}.d\vec{r}$ over every closed curve C is zero, then a is equal to

(a) –2 (b) –1 (c) 0 (d) 1

14. One of the integrating factor of the differential equation $(y^2 - 3xy)dx + (x^2 - xy)dy = 0$ is

(a) $1/(x^2y^2)$ (b) $1/(x^2y)$ (c) $1/(xy^2)$ (d) $1/(xy)$

15. Let C denote the boundary of the semi-circular disk $D = \left\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 1,\ y\ge 0\right\}$ and let $\varphi(x,y) = x^2 + y$ for $(x,y)\in D$. If $\hat{n}$ is the outward unit normal to C, then the integral $\int_C \left(\vec{\nabla}\varphi\right).\hat{n}ds$, evaluated counter-clockwise over C, is equal to

(a) 0 (b) $\pi - 2$ (c) π (d) $\pi + 2$

16. **(a)** Let $M = \begin{bmatrix} 1+i & 2i & i+3 \\ 0 & 1-i & 3i \\ 0 & 0 & i \end{bmatrix}$. Determine the eigen-values of the matrix $B = M^2 - 2M + I$. (9)

(b) Let N be a square matrix of order 2. If the determinant of N is equal to 9 and the sum of the diagonal entries of N is equal to 10, then determine the eigenvalues of N. (6)

17. **(a)** Using the variation of parameters, solve the differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2$, given that x and $\frac{1}{x}$ are two solutions of the corresponding homogeneous equations. (9)

(b) Find the real number α such that the differential equation

$\frac{d^2y}{dx^2} + 2(\alpha-1)(\alpha-3)\frac{dy}{dx} + (\alpha-2)y = 0$ has a solution $y(x) = a\cos(\beta x) + b\sin(\beta x)$ for some non-zero real numbers a, b, β. (6)

18. **(a)** Let a, b, c be non-zero real numbers such that $(a-b)^2 = 4ac$. Solve the differential equation.

$$a(x+\sqrt{2})^2\frac{d^2y}{dx^2} + b(x+\sqrt{2})\frac{dy}{dx} + cy = 0. \quad (9)$$

(b) Solve the differential equation $dx + (e^{y\sin y} - x)(y\cos y + \sin y)dy = 0$. (6)

19. Let $f(x, y) = x(x-2y^2)$ for $(x, y) \in \mathbb{R}^2$. Show that f has a local minimum at (0, 0) on every straight line through (0, 0). Is (0, 0) a critical point of f ? Find the discriminant of f at (0, 0). Does f have a local minimum at (0, 0) ? Justify your answers. (15)

20. **(a)** Find the finite volume enclosed by the paraboloids $z = 2 - x^2 - y^2$ and $z = x^2 + y^2$. (9)

(b) Let $f : [0,3] \to \mathbb{R}$ be a continuous function with $\int_0^3 f(x)dx = 3$.

Evaluate $\int_0^3 \left[xf(x) + \int_0^x f(t)dt \right] dx$. (6)

21. **(a)** Let S be the surface $\left\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + 2z = 2,\ z \geq 0\right\}$, and let $\hat{n}$ be the outward unit normal to S. If $\vec{F} = y\hat{i} + xz\hat{j} + \left(x^2 + y^2\right)\hat{k}$, then evaluate the integral $\iint_S \vec{F}.\hat{n}dS$. (9)

(b) Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $r = |\vec{r}|$. If a scalar field φ and a vector field $\vec{u}$ satisfy $\vec{\nabla}\varphi = \vec{\nabla} \times \vec{u} + f(r)\vec{r}$, where f is an arbitrary differentiable function , then show that $\nabla^2\varphi = rf'(r) + 3f(r)$. (6)

22. **(a)** Let D be the region bounded by the concentric spheres $S_1 : x^2 + y^2 + z^2 = a^2$ and $S_2 : x^2 + y^2 + z^2 = b^2$, where $a < b$. Let $\hat{n}$ be the unit normal to S_1 directed away from the origin. If $\nabla^2\phi = 0$ in D and $\phi = 0$ on S_2, then show that $\iiint_D \left|\vec{\nabla}\phi\right|^2 dV + \iint_{S_1} \phi\left(\vec{\nabla}\phi\right).\hat{n}dS = 0$. (9)

(b) Let C be the curve in $\mathbb{R}^3$ given by $x^2 + y^2 = a^2, z = 0$ traced counter-clockwise, and let $\vec{F} = x^2y^3\hat{i} + \hat{j} + z\hat{k}$. Using Stoke's theorem, evaluate $\int_C \vec{F}.d\vec{r}$. (6)

23. Let V be the subspace of $\mathbb{R}^4$ spanned by vector (1, 0, 1, 2), (2, 1, 3, 4) and (3, 1, 4, 6). Let $T : V \to \mathbb{R}^2$ be a linear transformation given by $T(x, y, z, t) = (x - y, z - t)$ for all $(x, y, z, t) \in V$. Find a basis for Null space of T and also a basis for range space of T. (15)

24. **(a)** Compute the double integral $I = \iint_D (x + 2y)dx\,dy$, where D is the region in the xy-plane bounded by the straight lines $y = x + 3, y = x - 3, y = -2x + 4$ and $y = -2x - 2$. (9)

(b) Evaluate $\int_0^{\pi/2} \left[\int_{\pi/2}^{\pi} \frac{\sin x}{x} dx \right] dy + \int_{\pi/2}^{\pi} \left[\int_y^{\pi} \frac{\sin x}{x} dx \right] dy$. (6)

25. **(a)** Does the series $\sum_{k=1}^{\infty} \frac{(-1)^k k + x^k}{k^2}$ converge uniformly for $x \in [-1, 1]$? Justify. (9)

(b) Suppose (f_n) is a sequence of real-valued functions defined on $\mathbb{R}$ and f is a real-valued function defined on $\mathbb{R}$ such that $|f_n(x) - f(x)| \le |a_n|$ for all $n \in \mathbb{N}$ and $a_n \to 0$ as $n \to \infty$. Must the sequence (f_n) be uniformly convergent on $\mathbb{R}$? Justify. (6)

26. **(a)** Suppose f is a real valued thrice differentiable function defined on $\mathbb{R}$ such that $f'''(x) > 0$ for all $x \in \mathbb{R}$. Using Taylor's formula, show that

$$f(x_2) - f(x_1) > (x_2 - x_1)\, f'\left(\frac{x_1 + x_2}{2}\right) \text{ for all } x_1, x_2 \text{ in } \mathbb{R} \text{ with } x_2 > x_1.$$ (9)

(b) Let (a_n) and (b_n) be sequences of real numbers such that $a_n \le a_{n+1} \le b_{n+1} \le b_n$ for all $n \in \mathbb{N}$. Must there exists a real number x such that $a_n \le x \le b_n$ for all $n \in \mathbb{N}$? Justify your answer. (6)

27. Let G be the group of all 2×2 matrices with real entries with respect to matrix multiplication. Let G_1 be the smallest subgroup of G containing $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and G_2 be the smallest subgroup of G containing $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Determine all elements of G_1 and find their orders. Determine all elements of G_2 and find their orders. Does there exist a one-to-one homomorphism from G_1 onto G_2? Justify. (15)

28. **(a)** Let p be a prime number and let $\mathbb{Z}$ be the ring of integers. If an ideal J of $\mathbb{Z}$ contains the set $p\mathbb{Z}$ properly, then show that $J = \mathbb{Z}$. (Here $p\mathbb{Z} = \{px : x \in \mathbb{Z}\}$). (9)

(b) Consider the ring $R = \{a + ib : a, b \in \mathbb{Z}\}$ with usual addition and multiplication. Find all invertible elements of R. (6)

29. **(a)** Suppose E is a non-empty subset of $\mathbb{R}$ which is bounded above, and let $\alpha = \sup E$. If E is closed, then show that $\alpha \in E$. If E is open, then show that $\alpha \notin E$. (9)

(b) Find all limit points of the set $E = \left\{n + \frac{1}{2m} : n, m \in \mathbb{N}\right\}$. (6)

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