PAPER : IIT-JAM 2009

MATHEMATICS-MA

(CODE–C)

Q.1-Q.15: Only one option is correct for each question. Each question carries (+6) marks for correct answer and (–2) marks for incorrect answer.

1. Let V be the vector space of all 6×6 real matrices over the field $\mathbb{R}$. Then the dimension of the subspace of V consisting of all symmetric matrices is

(a) 15 (b) 18 (c) 21 (d) 35

2. Let R be the ring of all functions from $\mathbb{R}$ to $\mathbb{R}$ under point-wise addition and multiplication. Let $I = \{ f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a bounded function} \}$, $J = \{ f : \mathbb{R} \to \mathbb{R} \mid f(3) = 0 \}$.

Then

(a) J is an ideal of R but I is not an ideal of R

(b) I is an ideal of R but J is not an ideal of R

(c) both I and J are ideals of R

(d) neither I nor J is an ideal of R

3. Which of the following sequences of functions is uniformly convergent on (0, 1) ?

(a) x^n (b) $\dfrac{n}{nx+1}$ (c) $\dfrac{x}{nx+1}$ (d) $\dfrac{1}{nx+1}$

4. Let $T : \mathbb{R}^4 \to \mathbb{R}^4$ be a linear transformation satisfy $T^3 + 3T^2 = 4I$, where I is the identity transformation, then the linear transformation $S = T^4 + 3T^3 - 4I$ is

(a) one-one but not onto (b) onto but not one-one

(c) invertible (d) non-invertible

5. The number of all subgroups of the group $(\mathbb{Z}_{60}, +)$ of integers modulo 60 is

(a) 2 (b) 10 (c) 12 (d) 60

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6. Let $a_n = \begin{cases} \dfrac{1}{3^n} & \text{if n is a prime,} \\ \dfrac{1}{4^n} & \text{if n is not a prime.} \end{cases}$

Then the radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$ is

(a) 4 (b) 3 (c) $\dfrac{1}{3}$ (d) $\dfrac{1}{4}$

7. The set of all limit points of the sequence $1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{16}, \frac{3}{16}, \frac{5}{16}, \frac{7}{16}, \frac{9}{16}, \ldots$ is

(a) [0, 1]

(b) (0, 1]

(c) the set of all rational numbers in [0, 1]

(d) the set of all rational numbers in [0, 1] of the form $\dfrac{m}{2^n}$ where m and n are integers

8. Let $F:\mathbb{R}\to\mathbb{R}$ be a continuous function and $a > 0$. Then the integral $\int_0^a\left[\int_0^x F(y)dy\right]dx$ equals

(a) $\int_0^a yF(y)dy$ (b) $\int_0^a (a-y)F(y)dy$ (c) $\int_0^a (y-a)F(y)dy$ (d) $\int_a^0 yF(y)dy$

9. The set of all positive values of a for which the series $\sum_{n=1}^{\infty}\left(\frac{1}{n}-\tan^{-1}\left(\frac{1}{n}\right)\right)^a$ converges, is

(a) $\left(0,\frac{1}{3}\right]$ (b) $\left(0,\frac{1}{3}\right)$ (c) $\left[\frac{1}{3},\infty\right)$ (d) $\left(\frac{1}{3},\infty\right)$

10. Let a be an non-zero real number. Then $\lim_{x\to a}\frac{1}{x^2-a^2}\int_a^x \sin(t^2)\,dt$ equals

(a) $\frac{1}{2a}\sin(a^2)$ (b) $\frac{1}{2a}\cos(a^2)$ (c) $-\frac{1}{2a}\sin(a^2)$ (d) $-\frac{1}{2a}\cos(a^2)$

11. Let $T(x,y,z)=xy^2+2z-x^2z^2$ be the temperature at the point (x,y,z). The unit vector in the direction in which the temperature decreases most rapidly at (1,0,–1) is

(a) $\frac{-1}{\sqrt{5}}\hat{i}+\frac{2}{\sqrt{5}}\hat{k}$ (b) $\frac{1}{\sqrt{5}}\hat{i}-\frac{2}{\sqrt{5}}\hat{k}$

(c) $\frac{2}{\sqrt{14}}\hat{i}+\frac{3}{\sqrt{14}}\hat{j}+\frac{1}{\sqrt{14}}\hat{k}$ (d) $-\left(\frac{2}{\sqrt{14}}\hat{i}+\frac{3}{\sqrt{14}}\hat{j}+\frac{1}{\sqrt{14}}\hat{k}\right)$

12. Consider the differential equation $2\cos(y^2)dx-xy\sin(y^2)dy=0$ then

(a) e^x is an integrating factor (b) e^{-x} is an integrating factor

(c) $3x$ is an integrating factor (d) x^3 is an integrating factor

13. Suppose $\vec{V}=p(x,y)\hat{i}+q(x,y)\hat{j}$ is a continuously differentiable vector field defined in a domain D in $\mathbb{R}^2$. Which one of the following statements is NOT equivalent to the remaining ones?

(a) There exists a function $\phi(x,y)$ such that $\frac{\partial\phi}{\partial x}=p(x,y)$ and $\frac{\partial\phi}{\partial y}=q(x,y)$ for all $(x,y)\in D$

(b) $\frac{\partial q}{\partial x}=\frac{\partial p}{\partial y}$ holds at all points of D

(c) $\oint_C \vec{V}.d\vec{r}=0$ for every piecewise smooth closed curve C in D

(d) The differential $pdx+qdy$ is exact in D

14. Let $f, g : [-1,1] \to \mathbb{R}$, $f(x) = x^3, g(x) = x^2 |x|$, then

(a) f and g are linear independent on [–1, 1]

(b) f and g are linearly dependent on [–1, 1]

(c) $f(x)g'(x) - f'(x)g(x)$ is NOT identically zero on [–1, 1].

(d) There exist continuous functions $p(x)$ and $q(x)$ such that f and g satisfy $y'' + py' + qy = 0$ on $[-1,1]$

15. The value of c for which there exists a twice differentiable vector field $\vec{F}$ with curl $\vec{F} = 2x\hat{i} - 7y\hat{j} + cz\hat{k}$ is

(a) 0 (b) 2 (c) 5 (d) 7

16. Consider A contains 100 cc of milk and container B contains 100 cc of water. 5 cc of the liquid in A is transferred to B, the mixture is thoroughly stirred and 5 cc of the mixture in B is transferred back into A. Each such two-way transfer is called a dilution. Let a_n be the percentage of water in container A after n such dilutions, with the understanding that $a_0 = 0$.

(a) Prove that $a_1 = \frac{100}{21}$ and that, in general, $a_n = \frac{100}{21} + \frac{19}{21} a_{n-1}$ for $n = 1, 2, 3, ...$ (6)

(b) Using (a) prove that $a_n = 50\left[1 - \left(\frac{19}{21}\right)^n\right]$ for $n = 1, 2, 3, ...$

Find $\lim_{n\to\infty} a_n$ and explain why the answer is intuitively obvious. (9)

17. **(a)** Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ be a non-negative function. Assume that for every $m \in \mathbb{N}$, the series $\sum_{n=1}^{\infty} f(m,n)$ is convergent and has sum a_m and further that the series $\sum_{m=1}^{\infty} a_m$ is also convergent and has sum L. Prove that for every n, the series $\sum_{m=1}^{\infty} f(m,n)$ is convergent and if we denote its sum by b_n then the series $\sum_{n=1}^{\infty} b_n$ is also convergent and has sum L. (9)

(b) Define $f : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ by

$$f(m,n) = \begin{cases} 0 & \text{if } n > m, \\ \frac{-1}{2^{m-n}} & \text{if } n < m, \\ 1 & \text{if } n = m. \end{cases}$$

Show that $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f(m,n) = 2$ and $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} f(m,n) = 0$. (6)

18. **(a)** Evaluate $\iint_R \cos(\max\{x^3, y^{3/2}\})\,dx\,dy$ where $R = [0, 1] \times [0, 1]$. (9)

(b) Let $S = \sqrt{1} + \sqrt{2} + \sqrt{3} + ... + \sqrt{10000}$ and $I = \int_0^{10000} \sqrt{x}\,dx$. Show that $I \le S \le I + 100$.

(6)

19. Let $D = \{(x, y) : x \ge 0, y \ge 0\}$. Let $f(x, y) = (x^2 + y^2)e^{-x-y}$ for $(x, y) \in D$.

Prove that f attains its maximum on D at two boundary points.

Deduce that $\dfrac{x^2 + y^2}{4} \le e^{x+y-2}$ for all $x \ge 0, y \ge 0$ (15)

20. **(a)** Let $a_1, b_1, a_2, b_2 \in \mathbb{R}$. Show that the condition $a_2 b_1 > 0$ is sufficient but not necessary for the system

$\dfrac{dx}{dt} = a_1 x + b_1 y$, $\dfrac{dy}{dt} = a_2 x + b_2 y$, to have two linearly independent solutions of the form $x = c_1 e^{\lambda_1 t}$, $y = d_1 e^{\lambda_1 t}$ and $x = c_2 e^{\lambda_2 t}, y = d_2 e^{\lambda_2 t}$ with $c_1, d_1, c_2, d_2, \lambda_1, \lambda_2 \in \mathbb{R}$ (9)

(b) Show that the differential equation representing the family of all straight lines which have an intercept of constant length L between the coordinate axes is $x\dfrac{dy}{dx} - y = \dfrac{L\dfrac{dy}{dx}}{\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}}$. (6)

21. Let A, B, $k > 0$. Solve the initial value problem $\dfrac{dy}{dx} - Ay + By^3 = 0, x > 0, y(0) = k$.

(a) Also show that $k < \sqrt{\dfrac{A}{B}}$, then the solution $y(x)$ is monotonically increasing on $(0, \infty)$ and tends to $\sqrt{\dfrac{A}{B}}$ as $x \to \infty$;

(b) if $k > \sqrt{\dfrac{A}{B}}$, then the solution $y(x)$ is monotonically decreasing on $(0, \infty)$ and tends to $\sqrt{\dfrac{A}{B}}$ as $x \to \infty$. (15)

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22. **(a)** Evaluate $\oint_C (3y^2+2z^2)dx+(6x-10z)ydy+(4xz-5y^2)dz$ along the portion from (1,0,1) to (3,4,5) of the curve *C*, which is the intersection of the surfaces $z^2=x^2+y^2$ and $z=y+1$. (6)

(b) A particle moves counterclockwise along the curve $3x^2+y^2=3$ from (1,0) to a point *P*, under the action of the force $\vec{F}(x,y)=\frac{x}{y}\hat{i}+\frac{y}{x}\hat{j}$. Prove that there are two possible locations of *P* such that the work done by $\vec{F}$ is 1. (9)

23. Verify Stokes theorem for the hemisphere $x^2+y^2+z^2=9, z\geq 0$ and the vector field $\vec{F}=\left(z^2-y\right)\hat{i}+\left(x-2yz\right)\hat{j}+\left(2xz-y^2\right)\hat{k}$. (15)

24. **(a)** Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be the linear transformation defined by $T(x,y,z)=(x+2y,x-z)$. Let $N(T)$ be the null space of *T* and $W=\{\vec{v}\in\mathbb{R}^3 \mid \vec{v}.\vec{u}=0 \text{ for all } \vec{u}\in N(T)\}$. Find a linear transformation $S:\mathbb{R}^2\to W$ such that *TS* = *I*, where *I* is the identity transformation on $\mathbb{R}^3$. (9)

(b) Suppose *A* is a real square matrix of odd order such that $A+A^T=0$. Prove that *A* is singular. (6)

25. **(a)** Find all pairs (a,b) of real numbers for which the system of equations

$x+3y=1,\ 4x+ay+z=0,\ 2x+3z=b$

has (i) a unique solution, (ii) infinitely many solution, (iii) no solution. (9)

(b) Let *A* be a $n\times n$ matrix such that $A^n=0$ and $A^{n-1}\neq 0$. Show that there exist a vector $v\in\mathbb{R}^n$ such that $[v, Av, ...A^{n-1}v]$ forms a basis for $\mathbb{R}^n$. (6)

26. **(a)** In which of the following pairs are the two groups isomorphic to each other? Justify your answers.

(i) $\mathbb{R}/\mathbb{Z}$ and S^1, where $\mathbb{R}$ is the additive group of real numbers and $S^1=\{z\in\mathbb{C}:|z|=1\}$ under complex multiplication.

(ii) $(\mathbb{Z},+)$ and $(\mathbb{Q},+)$. (9)

(b) Prove or disprove that if *G* is a finite abelian group of order *n*, and *k* is a positive integer which divides *n*, then *G* has at most one subgroup of order *k*. (6)

27. Let *I* and *J* be ideals of a ring *R*. Let *IJ* be the set of all possible sums $\sum_{i=1}^{n} a_i b_i$, where $a_i\in I, b_i\in J$ for i = 1, 2, ..., n and $n\in\mathbb{N}$.

(a) Prove that *IJ* is an ideal of *R* and $IJ\subseteq I\cap J$ (9)

(b) Is it true that $IJ=I\cap J$? Justify your answer. (6)

28. A sequence $\{f_n\}$ of functions defined on an interval I is said to be uniformly bounded on I if there exists some M such that $|f_n(x)| \le M$ for all $x \in I$ and for all $n \in N$.

(a) Prove that if a sequence of functions $\{f_n\}$ converges to a function f on I and $\{f_n\}$ is uniformly bounded on I, then f is bounded on I. (6)

(b) Suppose the sequences $\{f_n\}$ and $\{g_n\}$ of functions converge uniformly to f and g respectively on I and both are uniformly bounded on I. Prove that the product sequence $\{f_n g_n\}$ converges to fg uniformly on I. Show by an example that this may fail if only one of $\{f_n\}$ and $\{g_n\}$ is uniformly bounded on I. (9)

29. **(a)** Prove that if f is a real-valued function which is uniformly continuous on an interval (a, b), then f is bounded on (a, b). (9)

(b) Let f be a differentiable function on an interval (a, b). Assume that f' is bounded on (a, b). Prove that f is uniformly continuous on (a, b). (6)

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