PAPER : IIT-JAM 2010

MATHEMATICS-MA

(CODE–A)

Q.1-Q.15: Only one option is correct for each question. Each question carries (+6) marks for correct answer and (–2) marks for incorrect answer.

1. Which of the following conditions does NOT ensure the convergence of a real sequence $\{a_n\}$?

(a) $|a_n - a_{n+1}| \to 0$ as $n \to \infty$ (b) $\sum_{n=1}^{\infty} |a_n - a_{n+1}|$ is convergent

(c) $\sum_{n=1}^{\infty} n\, a_n$ is convergent (d) The sequences $\{a_{2n}\}, \{a_{2n+1}\}$ and $\{a_{3n}\}$ are convergent

2. The value of $\iint_G \frac{\ln(x^2+y^2)}{x^2+y^2} dx\, dy$, where $G = \{(x,y) \in \mathbb{R}^2; 1 \le x^2 + y^2 \le e^2\}$ is

(a) π (b) 2π (c) 3π (d) 4π

3. The number of elements of S_5 (the symmetric group on 5 letters) which are their own inverses equals

(a) 10 (b) 11 (c) 25 (d) 26

4. Let S be an infinite subset of $\mathbb{R}$ such that $S \cap \mathbb{Q} = \phi$. Which of the following statements is true?

(a) S must have a limit point which belongs to $\mathbb{Q}$

(b) S must have a limit point which belongs to $\mathbb{R} \setminus \mathbb{Q}$

(c) S cannot be a closed set in $\mathbb{R}$

(d) $\mathbb{R} \setminus S$ must have a limit point which belongs to S

5. Let $f : (1,4) \to \mathbb{R}$ be a uniformly continuous function and let $\{a_n\}$ be a Cauchy sequence in (1, 2).

Let $x_n = a_n^2 f(a_n^2)$ and $y_n = \frac{1}{1+a_n^2} f(a_n^2)$, for all $n \in N$. Which of the following statement is true?

(a) Both $\{x_n\}$ and $\{y_n\}$ must be Cauchy sequences in $\mathbb{R}$

(b) $\{x_n\}$ must be a Cauchy sequence in $\mathbb{R}$ but $\{y_n\}$ need not be a Cauchy sequence in $\mathbb{R}$

(c) $\{y_n\}$ must be a Cauchy sequence in $\mathbb{R}$ but $\{x_n\}$ need not be a Cauchy sequence in $\mathbb{R}$

(d) Neither $\{x_n\}$ nor $\{y_n\}$ needs to be a Cauchy sequence in $\mathbb{R}$

6. Let $\vec{F} = 2xyz\, e^{x^2} \hat{i} + z\, e^{x^2} \hat{j} + y e^{x^2} \hat{k}$ be the gradient of a scalar function. The value of $\int_L \vec{F}.d\vec{r}$ along the oriented path L from (0, 0, 0) to (1, 0, 2) and then to (1, 1, 2) is

(a) 0 (b) $2e$ (c) e (d) e^2

7. Let $\vec{F} = xy\hat{i} + y\hat{j} - yz\hat{k}$ denote the force field on a particle traversing the path L from (0, 0, 0) to (1, 1, 1) along the curve of intersection of the cylinder $y = x^2$ and the plane $z = x$. The work done by $\vec{F}$ is

(a) 0 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) 1

8. Let $\mathbb{R}[X]$ be the ring of real polynomials in the variables X. The number of ideals in the quotient ring $\mathbb{R}[X]/(X^2-3X+2)$ is

(a) 2 (b) 3 (c) 4 (d) 6

9. Consider the differential equation $\frac{dy}{dx} = ay - by^2$, where a, $b > 0$ and $y(0) = y_0$. As $x \to +\infty$, then solution $y(x)$ tends to

(a) 0 (b) a/b (c) b/a (d) y_0

10. Consider the differential equation $(x+y+1)dx + (2x+2y+1)dy = 0$. Which of the following statements is true?

(a) The differential equation is linear
(b) The differential equation is exact
(c) e^{x+y} is an integrating factor of the differential equation
(d) A suitable substitution transforms the differentiable equation to the variables separable form

11. Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that $T\big((1,2)\big) = (2,3)$ and $T\big((0,1)\big) = (1,4)$. Then $T(5,6)$ is

(a) (6, –1) (b) (–6, 1) (c) (–1, 6) (d) (1, –6)

12. The number of 2 × 2 matrices over $\mathbb{Z}_3$ (the field with three elements) with determinant 1 is

(a) 24 (b) 60 (c) 20 (d) 30

13. The radius of convergence of the power series $\sum_{n=0}^{\infty} a_n z^{n^2}$, where $a_0 = 1, a_n = 3^{-n} a_{n-1}$ for $n \in \mathbf{N}$, *is*

(a) 0 (b) $\sqrt{3}$ (c) 3 (d) ∞

14. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix with respect to standard basis $\{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ is $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Then T

(a) maps the subspace spanned by e_1 and e_2 into itself
(b) has distinct eigenvalues
(c) has eigenvectors that span $\mathbb{R}^3$
(d) has a non-zero null space

15. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix with respect to the standard basis of $\mathbb{R}^3$

is $\begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix}$, where a, b, c are real numbers not all zero. Then T

(a) is one to one
(b) is onto
(c) does not map any line passes through the origin onto itself
(d) has rank 1

16. **(a)** Obtain the general solution of the following system of differential equations:

$$\frac{dx}{dt} = x + 2y, \quad \frac{dy}{dt} = 4x - y + e^{3t}$$

(9)

(b) Find the curve passing through $\left(\frac{1}{2}, 0\right)$ and having slope at (x, y) given by differential equation $2(1+y^2)dx + (2x - \tan^{-1} y)dy = 0$.

(6)

17. **(a)** Find the volume of the region in the first octant bounded by the surfaces $x = 0$, $y = x$, $y = 2 - x^2$, $z = 0$ and $z = x^2$. (6)

(b) Suppose $f:\mathbb{R} \to \mathbb{R}$ is a non-constant continuous function satisfying $f(x+y) = f(x)\,f(y)$ for all $x, y \in \mathbb{R}$.

(*i*) Show that $f(x) \neq 0$ for all $x \in \mathbb{R}$.

(*ii*) Show that $f(x) > 0$ for all $x \in \mathbb{R}$.

(*iii*) Show that there exists $\beta \in \mathbb{R}$ such that $f(x) = \beta^x$ for all $x \in \mathbb{R}$. (9)

18. **(a)** Let $f(x)$ and $g(x)$ be real valued functions continuous in $[a, b]$, differentiable in (a, b) and let $g'(x) \neq 0$ for all $x \in (a,b)$. Show that there exists $c \in (a,b)$ such that $\dfrac{f(c) - f(a)}{g(b) - g(c)} = \dfrac{f'(c)}{g'(c)}$. (9)

(b) Let $0 < \lambda < 4$ and let $\{a_n\}$ be a sequence of positive real numbers satisfying $a_{n+1} = \lambda a_n^2(1 - a_n)$ for $n \in \mathbb{N}$. Prove that $\lim\limits_{n\to\infty} a_n$ exists and determine this limit. (6)

19. Let G be an open subset of $\mathbb{R}$.

(a) If $0 \notin G$, then show that $H = \{xy : x, y \in G\}$ is an open subset of $\mathbb{R}$. (9)

(b) If $0 \in G$ and if $x + y \in G$ for all $x, y \in G$, then show that $G = \mathbb{R}$ (6)

20. Let $p(x)$ be a non-constant polynomial with real coefficients such that $p(x) \neq 0$ for all $x \in \mathbb{R}$.

Define $f(x) = \dfrac{1}{p(x)}$ for all $x \in \mathbb{R}$. Prove that

(i) For each $\varepsilon > 0$, there exists a > 0 such that $|f(x)| < \varepsilon$ for all $x \in R$ satisfying $|x| > a$, and

(ii) $f : \mathbb{R} \to \mathbb{R}$ is a uniformly continuous function. (15)

21. **(a)** Let $M(K)$ and $m(k)$ denote respectively the absolute maximum and the absolute minimum values of $x^3 + 9x^2 - 21x + k$ in the closed interval $[-10, 2]$. Find all the real values of k for which $|M(k)| = |m(k)|$. (6)

(b) Let $\alpha_1 = 0, \beta_1 = 1; \alpha_2 = 1, \beta_2 = 1$, and for $n \geq 3$,

$\alpha_n = \alpha_{n-1} + 2\alpha_{n-2}$,

$\beta_n = \beta_{n-1} + 2\beta_{n-2}$.

Prove that, for $n \in \mathbb{N}$

(i) $\beta_n = 2\alpha_n + (-1)^{n-1}$

(ii) $\alpha_n + \beta_n = 2^{n-1}$

Deduce that $\lim\limits_{n\to\infty} \dfrac{\alpha_n a + \beta_n b}{2^{n-1}} = \dfrac{a+2b}{3}$ for any $a, b \in \mathbb{R}$. (9)

22. **(a)** Let $f(x,y) = \alpha x^2 + xy + \beta y^2, \alpha \neq 0, \beta \neq 0, 4\alpha\beta \neq 1$. Find sufficient conditions on (α, β) such that $(0,0)$ is (9)

(i) a point of local maxima of $f(x,y)$

(ii) a point of local minima of $f(x,y)$

(iii) a saddle point of $f(x,y)$

(b) Find the derivative of $f(x,y,z) = 7x^3 - x^2 z - z^2 + 28y$ at the point A = (1, –1, 0) along the unit vector $\frac{1}{7}(6\hat{i} - 2\hat{j} + 3\hat{k})$. What is the unit vector along which f decreases most rapidly at A? Also, find the rate of this decreases. (6)

23. Using $x = e^u$, transform the differential equation

$$x^2 \frac{d^2 y}{dx^2} + 4x\frac{dy}{dx} + 2y = \cos x$$

to a second order differential equation with constant coefficients. Obtain the general solution of the transformed differential equation. (15)

24. Let G be a group and let $A(G)$ denote the set of all automorphism of G, i.e. all one-to-one, onto, group homomorphisms from G to G. An automorphism $f : G \to G$ of the form $f(x) = axa^{-1}, x \in G$ (for some $a \in G$) is called an inner automorphism. Let $I(G)$ denote the set of all inner automorphism of G.

(a) Show that $A(G)$ is a group under composition of functions and that $I(G)$ is a normal subgroup of $A(G)$. (9)

(b) Show that $I(G)$ is isomorphic to $G/Z(G)$, where

$Z(G) = \{g \in G : xg = gx \text{ for all } x \in G\}$ is the centre of G. (6)

25. **(a)** Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T^2(v) = -v$ for all $v \in \mathbb{R}^2$. (6)

(b) Let V be a real n-dimensional vector space and let $T : V \to V$ be a linear transformation satisfying $T^2(v) = -v$ for all $v \in V$.

(i) Show that n is even.

(ii) Use T to make V into complex vector space such that the multiplication by complex numbers extends the multiplication by real numbers.

(iii) Show that, with respect to the complex vector space structure on V obtained in (ii), $T : V \to V$ is a complex linear transformation. (9)

26. Let W be the region bounded by the planes $x = 0, y = 0, y = 3, z = 0$ and $x + 2z = 6$. Let S be the boundary of this region. Using Gauss' divergence theorem, evaluate $\iint_S F.\hat{n} dS$, where $\vec{F} = 2xy\hat{i} + yz^2\hat{j} + xz\hat{k}$ and $\hat{n}$ is the outward unit normal vector to S. (15)

27. **(a)** Using strokes' theorem evaluate the line integral $\int_L \left(y\hat{i} + z\hat{j} + x\hat{k}\right).d\vec{r}$, where L is the intersection of $x^2 + y^2 + z^2 = 1$ and $x + y = 0$ traversed in the clockwise direction when viewed from the point (1, 1, 0). (9)

(b) Change the order of integration in the integral $\int_0^1 \int_{x-1}^{\sqrt{1-x^2}} f(x, y) dy\, dx$. (6)

28. In a group G, $x \in G$ is said to be conjugate to $y \in G$, written $x \sim y$, if there exists $z \in G$ such that $x = zyz^{-1}$.

(a) Show that ~ is an equivalence relation on G. Show that a subgroup N of G is a normal subgroup of G if and only if N is a union of equivalence classes of ~. (6)

(b) Consider the group of all non-singular 3×3 real matrices under matrix multiplication. Show that $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 3 & 0 & 4 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (i.e. the two matrices are conjugate). (9)

29. Let S denote the commutative ring of all continuous real valued functions on [0, 1], under pointwise addition and multiplication. For $a \in [0,1]$, let $M_a = \{ f \in S \mid f(a) = 0 \}$.

(a) Show that M_a is an ideal in S. (6)

(b) Show that M_a is a maximal ideal in S. (9)

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