PAPER : IIT-JAM 2017

MATHEMATICS-MA

SECTION-A

Multiple Choice Questions (MCQ)

Q.1 – Q.10 carry ONE mark each.

1. Let $f:\mathbb{R}\to\mathbb{R}$ be a twice differentiable function. If $g(u,v)=f\left(u^2-v^2\right)$, then

$$\frac{\partial^2 g}{\partial u^2}+\frac{\partial^2 g}{\partial v^2}=$$

(a) $4\left(u^2-v^2\right)f''\left(u^2-v^2\right)$ (b) $4\left(u^2+v^2\right)f''\left(u^2-v^2\right)$

(c) $2f'\left(u^2-v^2\right)+4\left(u^2-v^2\right)f''\left(u^2-v^2\right)$ (d) $2\left(u-v\right)^2 f''\left(u^2-v^2\right)$

2. Let

$$f(x)=\frac{x+|x|(1+x)}{x}\sin\left(\frac{1}{x}\right),\quad x\neq 0.$$

Write $L=\lim_{x\to 0^-}f(x)$ and $R=\lim_{x\to 0^+}f(x)$. Then which one of the following is TRUE?

(a) L exists but R does not exist (b) L does not exist but R exists

(c) Both L and R exist (d) Neither L nor R exists

3. Let $f_1(x),\ f_2(x),\ g_1(x),\ g_2(x)$ be differentiable functions on $\mathbb{R}$. Let $F(x)=\begin{vmatrix} f_1(x) & f_2(x)\\ g_1(x) & g_2(x)\end{vmatrix}$ be the determinant of the matrix $\begin{bmatrix} f_1(x) & f_2(x)\\ g_1(x) & g_2(x)\end{bmatrix}$. Then $F'(x)$ is equal to

(a) $\begin{vmatrix} f_1'(x) & f_2'(x)\\ g_1(x) & g_2(x)\end{vmatrix}+\begin{vmatrix} f_1(x) & g_1'(x)\\ f_2'(x) & g_2(x)\end{vmatrix}$ (b) $\begin{vmatrix} f_1'(x) & f_2'(x)\\ g_1(x) & g_2(x)\end{vmatrix}+\begin{vmatrix} f_1(x) & g_1'(x)\\ f_2(x) & g_2'(x)\end{vmatrix}$

(c) $\begin{vmatrix} f_1'(x) & f_2'(x)\\ g_1(x) & g_2(x)\end{vmatrix}-\begin{vmatrix} f_1(x) & g_1'(x)\\ f_2(x) & g_2'(x)\end{vmatrix}$ (d) $\begin{vmatrix} f_1'(x) & f_2'(x)\\ g_1'(x) & g_2'(x)\end{vmatrix}$

4. If $\lim_{T\to\infty}\int_0^T e^{-x^2}dx=\frac{\sqrt{\pi}}{2}$, then

$$\lim_{T\to\infty}\int_0^T x^2e^{-x^2}dx=$$

(a) $\frac{\sqrt{\pi}}{4}$ (b) $\frac{\sqrt{\pi}}{2}$ (c) $\sqrt{2\pi}$ (d) $2\sqrt{\pi}$

5. If $f(x)=\begin{cases}1+x & \text{if } x<0\\ (1-x)(px+q) & \text{if } x\geq 0\end{cases}$

satisfies the assumptions of Rolle's theorem in the interval [–1, 1], then the ordered pair (p, q) is

(a) (2, –1) (b) (–2, –1) (c) (–2, 1) (d) (2, 1)

6. $\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx =$

(a) $\frac{1+\cos 1}{2}$ (b) $1-\cos 1$ (c) $1+\cos 1$ (d) $\frac{1-\cos 1}{2}$

7. The number of generators of the additive group $\mathbb{Z}_{36}$ is equal to

(a) 6 (b) 12 (c) 18 (d) 36

8. Consider the function $f(x, y) = 5 - 4\sin x + y^2$ for $0 < x < 2\pi$ and $y \in \mathbb{R}$. The set of critical points of $f(x, y)$ consists of

(a) a point of local maximum and a point of local minimum
(b) a point of local maximum and a saddle point
(c) a point of local maximum, a point of local minimum and a saddle point
(d) a point of local minimum and a saddle point

9. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that φ' is strictly increasing with $\varphi'(1) = 0$. Let α and β denote the minimum and maximum values of $\varphi(x)$ on the interval [2, 3], respectively. then which one of the following is TRUE?

(a) $\beta = \varphi(3)$ (b) $\alpha = \varphi(2.5)$ (c) $\beta = \varphi(2.5)$ (d) $\alpha = \varphi(3)$

10. $\lim_{n\to\infty} \frac{\pi}{n} \sum_{k=1}^{n} \sin\left(\frac{\pi}{2} + \frac{5\pi}{2} \cdot \frac{k}{n}\right) =$

(a) $\frac{2\pi}{5}$ (b) $\frac{5}{2}$ (c) $\frac{2}{5}$ (d) $\frac{5\pi}{2}$

Q.11 – Q.30 carry TWO marks each.

11. Let $y(x)$ be the solution of the differential equation

$$(xy + y + e^{-x})dx + (x + e^{-x})dy = 0$$

satisfying $y(0) = 1$. Then $y(-1)$ is equal to

(a) $\frac{e}{e-1}$ (b) $\frac{2e}{e-1}$ (c) $\frac{e}{1-e}$ (d) 0

12. Let $f : \mathbb{R} \to [0, \infty)$ be a continuous function. Then which one of the following is NOT TRUE?

(a) There exists $x \in \mathbb{R}$ such that $f(x) = \frac{f(0) + f(1)}{2}$

(b) There exists $x \in \mathbb{R}$ such that $f(x) = \sqrt{f(-1) f(1)}$

(c) There exists $x \in \mathbb{R}$ such that $f(x) = \int_{-1}^{1} f(t)\, dt$

(d) There exists $x \in \mathbb{R}$ such that $f(x) = \int_0^1 f(t)\, dt$

13. Let $M = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ 0 & 1 \end{bmatrix}$ and $x = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$. Then

$$\lim_{n\to\infty} M^n x$$

(a) does not exist (b) is $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (c) is $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$ (d) is $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$

14. Let $0 < a_1 < b_1$. For $n \geq 1$, define

$$a_{n+1} = \sqrt{a_n b_n} \text{ and } b_{n+1} = \frac{a_n + b_n}{2}$$

Then which one of the followings is NOT TRUE?

(a) Both $\{a_n\}$ and $\{b_n\}$ converge, but the limits are not equal

(b) Both $\{a_n\}$ and $\{b_n\}$ converge and the limits are equal

(c) $\{b_n\}$ is a decreasing sequence (d) $\{a_n\}$ is an increasing sequence

15. The line integral of the vector field

$$\vec{F} = zx\,\hat{i} + xy\,\hat{j} + yz\,\hat{k}$$

along the boundary of the triangle with vertices (1, 0, 0), (0, 1, 0) and (0, 0, 1), oriented anticlockwise, when viewed from the point (2, 2, 2), is

(a) $\frac{-1}{2}$ (b) -2 (c) $\frac{1}{2}$ (d) 2

16. The flux of the vector field

$$\vec{F} = \left(2\pi x + \frac{2x^2y^2}{\pi}\right)\hat{i} + \left(2\pi xy - \frac{4y}{\pi}\right)\hat{j}$$

along the outward normal, across the ellipse $x^2 + 16y^2 = 4$ is equal to

(a) $4\pi^2 - 2$ (b) $2\pi^2 - 4$ (c) $\pi^2 - 2$ (d) 2π

17. $\lim_{x\to\infty} \frac{1}{\sqrt{n}}\left(\frac{1}{\sqrt{3}+\sqrt{6}} + \frac{1}{\sqrt{6}+\sqrt{9}} + \cdots + \frac{1}{\sqrt{3n}+\sqrt{3n+3}}\right) =$

(a) $1+\sqrt{3}$ (b) $\sqrt{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{1+\sqrt{3}}$

18. $\sum_{n=1}^{\infty} \tan^{-1}\frac{2}{n^2} =$

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$ (c) $\frac{3\pi}{4}$ (d) π

19. Let S be an infinite subset of $\mathbb{R}$ such that $S \setminus \{a\}$ is compact for some $a \in S$. Then which one of the following is TRUE?

(a) S is a connected set (b) S contains no limit points

(c) S is a union of open intervals

(d) Every sequence in S has a subsequence converging to an element in S

20. A particular integral of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} = e^{2x}\sin x$$

is

(a) $\frac{e^{2x}}{10}(3\cos x - 2\sin x)$ (b) $-\frac{e^{2x}}{10}(3\cos x - 2\sin x)$

(c) $-\frac{e^{2x}}{5}(2\cos x + \sin x)$ (d) $\frac{e^{2x}}{5}(2\cos x - \sin x)$

21. The area of the surface $z = \frac{xy}{3}$ intercepted by the cylinder $x^2 + y^2 \le 16$ lies in the interval

(a) $(20\pi, 22\pi]$ (b) $(22\pi, 24\pi]$ (c) $(24\pi, 26\pi]$ (d) $(26\pi, 28\pi]$

22. Let P_3 denote the real vector space of all polynomials with real coefficients of degree at most 3. Consider the map $T : P_3 \to P_3$ given by $T\left(p(x)\right) = p''(x) + p(x)$. Then

(a) T is neither one-one nor onto (b) T is both one-one and onto

(c) T is one-one but not onto (d) T is onto but not one-one

23. Let M be the set of all invertible 5 × 5 matrices with entries 0 and 1. For each $M \in \mathrm{M}$, let $n_1(M)$ and $n_0(M)$ denote the number of 1's and 0's in M, respectively. Then

$$\min_{M \in \mathrm{M}} \left| n_1(M) - n_0(M) \right| =$$

(a) 1 (b) 3 (c) 5 (d) 15

24. The interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{1}{(-3)^{n+2}} \frac{(4x-12)^n}{n^2+1}$$

is

(a) $\frac{10}{4} \le x < \frac{14}{4}$ (b) $\frac{9}{4} \le x < \frac{15}{4}$ (c) $\frac{10}{4} \le x \le \frac{14}{4}$ (d) $\frac{9}{4} \le x \le \frac{15}{4}$

25. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(2) = 2$ and

$$\left| f(x) - f(y) \right| \le 5\left(|x-y|\right)^{3/2}$$

for all $x \in \mathbb{R}, y \in \mathbb{R}$. Let $g(x) = x^3 f(x)$. Then $g'(2) =$

(a) 5 (b) $\frac{15}{2}$ (c) 12 (d) 24

26. Let $f(x, y) = \frac{x^2 y}{x^2 + y^2}$ for $(x, y) \ne (0,0)$. Then

(a) $\frac{\partial f}{\partial x}$ and f are bounded (b) $\frac{\partial f}{\partial x}$ is bounded and f is unbounded

(c) $\frac{\partial f}{\partial x}$ is unbounded and f is bounded (d) $\frac{\partial f}{\partial x}$ and f are unbounded

27. For $a > 0, b > 0$, let $\vec{F} = \frac{x\hat{i} - y\hat{j}}{b^2x^2 + a^2y^2}$ be a planar vector field. Let $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = a^2 + b^2\}$ be the circle oriented anti-clockwise. Then $\oint_C \vec{F} \cdot d\vec{r} =$

(a) $\frac{2\pi}{ab}$ (b) 2π (c) $2\pi ab$ (d) 0

28. Which one of the followings is TRUE?
(a) Every sequence that has a convergent subsequence is a Cauchy sequence
(b) Every sequence that has a convergent subsequence is a bounded sequence
(c) The sequence $\{\sin n\}$ has a convergent subsequence
(d) The sequence $\left\{n\cos\frac{1}{n}\right\}$ has a convergent subsequence

29. Let $\vec{F} = (3 + 2xy)\hat{i} + (x^2 - 3y^2)\hat{j}$ and let L be the curve

$$\vec{r}(t) = e^t \sin t\,\hat{i} + e^t \cos t\,\hat{j}, \qquad 0 \le t \le \pi.$$

Then

$$\int_L \vec{F} \cdot d\vec{r} =$$

(a) $e^{-3\pi} + 1$ (b) $e^{-6\pi} + 2$ (c) $e^{6\pi} + 2$ (d) $e^{3\pi} + 1$

30. The flux of $\vec{F} = y\,\hat{i} - x\,\hat{j} + z^2\hat{k}$ along the outward normal, across the surface of the solid

$$\left\{(x, y, z) \in \mathbb{R}^3 \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le \sqrt{2 - x^2 - y^2}\right\}$$

is equal to

(a) $\frac{2}{3}$ (b) $\frac{5}{3}$ (c) $\frac{8}{3}$ (d) $\frac{4}{3}$

SECTION-B

Multiple Select Questions (MSQ)

Q.31 – Q.40 carry TWO marks each.

31. Let $f:\mathbb{R}^2 \to \mathbb{R}$ be a function. Then which of the following statements is/are TRUE?

(a) If f is differentiable at (0, 0), then all directional derivatives of f exist at (0, 0)

(b) If all directional derivatives of f exist at (0, 0), then f is differentiable at (0, 0)

(c) If all directional derivatives of f exist at (0, 0), then f is continuous at (0, 0)

(d) If the partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous in a disc centered at (0, 0), then f is differentiable at (0, 0)

32. Let M be an $n\times n$ matrix with real entries such that $M^3 = I$. Suppose that $Mv \neq v$ for any non zero vector v. Then which of the following statements is/are TRUE?

(a) M has real eigenvalues
(b) $M+M^{-1}$ has real eigenvalues
(c) n is divisible by 2
(d) n is divisible by 3

33. Let $k_2\ell \in \mathbb{R}$ be such that every solution of

$$\frac{d^2y}{dx^2}+2k\frac{dy}{dx}+\ell y=0$$

satisfies $\lim\limits_{x\to\infty} y(x)=0$. Then

(a) $3k^2+\ell<0$ and $k>0$
(b) $k^2+\ell>0$ and $k<0$
(c) $k^2-\ell\le 0$ and $k>0$
(d) $k^2-\ell>0, k>0$ and $\ell>0$

34. The volume of the solid $\left\{(x,y,z)\in\mathbb{R}^3 \middle| 1\le x\le 2,\quad 0\le y\le \frac{2}{x},\quad 0\le z\le x\right\}$ is expressible as

(a) $\int_1^2\int_0^{2/x}\int_0^x dz\,dy\,dx$
(b) $\int_1^2\int_0^x\int_0^{2/x} dy\,dz\,dx$
(c) $\int_0^2\int_1^2\int_0^{2/x} dy\,dx\,dz$
(d) $\int_0^2\int_{\max[2,1]}^2\int_0^{2/x} dy\,dx\,dz$

35. Let $y(x)$ be the solution of the differential equation

$$\frac{dy}{dx}=(y-1)(y-3)$$

satisfying the condition $y(0)=2$. Then which of the followings is/are TRUE?

(a) The function $y(x)$ is not bounded above
(b) The function $y(x)$ is bounded
(c) $\lim\limits_{x\to+\infty} y(x)=1$
(d) $\lim\limits_{x\to-\infty} y(x)=3$

36. Let G be a group of order 20 in which the conjugacy classes have sizes 1, 4, 5, 5, 5. Then which of the followings is/are TRUE?

(a) G contains a normal subgroup of order 5
(b) G contains a non-normal subgroup of order 5
(c) G contains a subgroup of order 10
(d) G contains a normal subgroup of order 4

37. Let $\{x_n\}$ be a real sequence such that $7x_{n+1} = x_n^3 + 6$ for $n \geq 1$. Then which of the following statements is/are TRUE?

(a) If $x_1 = \frac{1}{2}$, then $\{x_n\}$ converges to 1
(b) If $x_1 = \frac{1}{2}$, then $\{x_n\}$ converges to 2
(c) If $x_1 = \frac{3}{2}$, then $\{x_n\}$ converges to 1
(d) If $x_1 = \frac{3}{2}$, then $\{x_n\}$ converges to –3

38. For $\alpha, \beta \in \mathbb{R}$, define the map $\varphi_{\alpha,\beta} : \mathbb{R} \to \mathbb{R}$ by $\varphi_{\alpha,\beta}(x) = \alpha x + \beta$. Let $G = \{\varphi_{\alpha,\beta} \mid (\alpha,\beta) \in \mathbb{R}^2\}$

For $f, g \in G$, define $g \circ f \in G$ by $(g \circ f)(x) = g(f(x))$. Then which of the following statements is/are TRUE?

(a) The binary operation $\circ$ is associative
(b) The binary operation $\circ$ is commutative
(c) For every $(\alpha,\beta) \in \mathbb{R}^2, \alpha \neq 0$ there exists $(a,b) \in \mathbb{R}^2$ such that $\varphi_{\alpha,\beta} \circ \varphi_{a,b} = \varphi_{1,0}$
(d) $(G, \circ)$ is group

39. If X and Y are $n \times n$ matrices with real entries, then which of the following is/are TRUE?

(a) If $P^{-1}XP$ is diagonal for some real invertible matrix P, then there exists a basis for $\mathbb{R}^n$ consisting of eigenvectors of X
(b) If X is diagonal with distinct diagonal entries and XY = YX, then Y is also diagonal
(c) If X^2 is diagonal, then X is diagonal
(d) If X is diagonal and XY = YX for all Y, then $X = \lambda I$ for some $\lambda \in \mathbb{R}$

40. Let S be the set of all rational numbers in (0, 1). Then which of the following statements is/are TRUE?

(a) S is a closed subset of $\mathbb{R}$
(b) S is not a closed subset of $\mathbb{R}$
(c) S is an open subset of $\mathbb{R}$
(d) Every $x \in (0,1) \setminus S$ is a limit point of S

SECTION-C

Numerical Answer Type (NAT)

Q.41 – Q.50 carry ONE mark each.

41. Let G be a subgroup of $GL_2(\mathbb{R})$ generated by $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$. Then the order of G is __________

42. Let P be the point on the surface $z = \sqrt{x^2 + y^2}$ closest to the point (4, 2, 0). Then the square of the distance between the origin and P is ____________

43. Consider the permutations $\sigma = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 4\,5\,3\,7\,8\,6\,1\,2 \end{pmatrix}$ and $\tau = \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ 4\,5\,3\,1\,7\,6\,8\,2 \end{pmatrix}$ in S_8. The number of $\eta \in S_8$ such that $\eta^{-1}\sigma\eta = \tau$ is equal to ________

44. For $x > 0$, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to *x*. Then

$$\lim_{x \to 0^+} x\left(\left\lfloor \frac{1}{x} \right\rfloor + \left\lfloor \frac{2}{x} \right\rfloor + \cdots + \left\lfloor \frac{10}{x} \right\rfloor\right) = \_\_\_\_\_\_\_\_\_\_$$

45. The number of subgroups of $\mathbb{Z}_7 \times \mathbb{Z}_7$ of order 7 is __________

46. Let $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Let M be the matrix whose columns are $v_1, v_2, 2v_1 - v_2, v_1 + 2v_2$ in that order. Then the number of linearly independent solutions of the homogeneous system of linear equations $Mx = 0$ is __________

47. $\left(\int_0^1 x^4 (1-x)^5 \, dx\right)^{-1} =$ __________

48. Let P be a 7 × 7 matrix of rank 4 with real entries. Let $a \in \mathbb{R}^7$ be a column vector. Then the rank of $P + aa^T$ is at least __________

49. If the orthogonal trajectories of the family of ellipses $x^2 + 2y^2 = c_1, c_1 > 0$, are given by $y = c_2 x^a, c_2 \in \mathbb{R}$, then $\alpha =$ __________

50. $\frac{1}{2\pi}\left(\frac{\pi^3}{1!3} - \frac{\pi^5}{3!5} + \frac{\pi^7}{5!7} - \cdots + \frac{(-1)^{n-1}\pi^{2n+1}}{(2n-1)!(2n+1)} + \cdots\right) =$ __________

Q.51 – Q.60 carry TWO marks each.

51. The radius of convergence of the power series, $\sum_0^\infty n! x^{n^2}$ is __________

52. Let $f(x) = \frac{\sin \pi x}{\pi \sin x}, x \in (0, \pi)$, and let $x_0 \in (0, \pi)$ be such that $f'(x_0) = 0$. Then

$$\left(f(x_0)\right)^2 \left(1 + \left(\pi^2 - 1\right)\sin^2 x_0\right) = \_\_\_\_\_\_\_\_\_\_$$

53. Let $\alpha, \beta, \gamma, \delta$ be the eigenvalues of the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Then $a^2 + \beta^2 + \gamma^2 + \delta^2 =$ __________

54. Let $a_n = \sqrt{n}, n \geq 1$, and let $s_n = a_1 + a_2 + \cdots + a_n$. Then

$$\lim_{x \to \infty} \left(\frac{a_n / S_n}{-\ln(1 - a_n / S_n)}\right) = \_\_\_\_\_\_\_\_\_\_$$

55. For $x > 1$, let $f(x) = \int_1^x \left(\sqrt{\log t} - \frac{1}{2}\log\sqrt{t}\right) dt$

The number of tangents to the curve $y = f(x)$ parallel to the line $x + y = 0$ is __________

56. For a real number x, define $\lceil x \rceil$ to be the smallest integer greater than or equal to x. Then

$$\int_0^1\int_0^1\int_0^1 \left(\lceil x \rceil + \lceil y \rceil + \lceil z \rceil\right) dx\, dy\, dz = \_\_\_\_\_\_\_\_\_\_\_\_$$

57. If $y(x) = \int_{\sqrt{x}}^{x} \frac{e^t}{t} dt,\ x > 0$

then $y'(1) =$ ____________

58. Let $y(x),\ x > 0$ be the solution of the differential equation

$$x^2 \frac{d^2 y}{dx^2} + 5x\frac{dy}{dx} + 4y = 0$$

satisfying the conditions $y(1) = 1$ and $y'(1) = 0$. Then the value of $e^2 y(e)$ is ____________

59. The maximum order of a permutation σ in the symmetric group S_{10} is ____________

60. Let T be the smallest positive real number such that the tangent to the helix

$$\cos t\ \hat{i} + \sin t\ \hat{j} + \frac{t}{\sqrt{2}}\hat{k}$$

at $t = T$ is orthogonal to the tangent at $t = 0$. Then the line integral of $\vec{F} = x\hat{j} - y\hat{i}$ along the section of the helix from $t = 0$ to $t = T$ is __________________

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