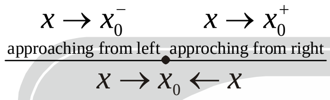
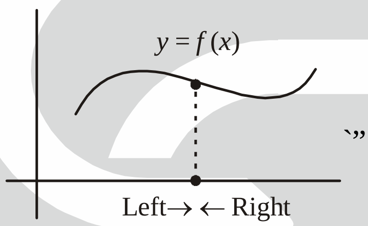
Limits and Continuity

Chapter 2

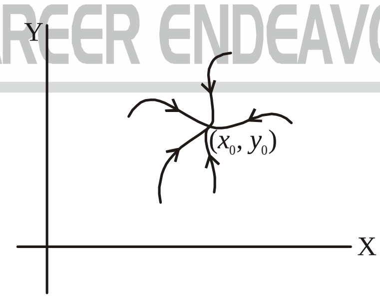
Limits:- Recall that in the case of limit of functions of one variable, we say $\lim_{x \to x_0} f(x)$ exist if both $\lim_{x \to x_0^+} f(x)$ and $\lim_{x \to x_0^-} f(x)$ exist and equal. Here $x \to x_0^+$ and $x \to x_0^-$ reflecting the fact that there are only two directions from which *x* can approach to x_0, the right of *x* or left of *x*.



So limit is said to be exist if value of limit (if exist) along both possible path should be same.



But for a function of two variables, there are infinitely many directions in the 2-dimensional space along which (*x*, *y*) can approach to (x_0, y_0), because there are infinite number of possible curves along which one point can approach another.



So we say that $\lim_{(x,y) \to (x_0 y_0)}$ exist, if limit exists along all the possible paths and has same value.

Limits Along Curves:

If *C* is a smooth parameteric curve in 2-dimensional space that is represented by the equations $x = x(t),\ y = y(t)$, then

$$\lim_{\substack{(x,y) \to (x_0, y_0) \\ \text{along C}}} f(x, y) = \lim_{t \to t_0} f(x(t), y(t)) \quad \text{where } t_0 \text{ is parametricvalue for } (x_0, y_0)$$

Example

$$f(x,y) = -\frac{xy}{x^2+y^2}$$

Find the limit of $f(x,y)$ at origin through the following curves?

(a) x-axis (b) y-axis (c) the line $y = mx$ (d) the parabola $y = x^2$

Soln. (a) The x-axis has parametric equations $x = t,\ y = 0$, so $\lim\limits_{\substack{(x,y)\to(0,0)\\ (\text{along } y=0)}} f(x,y) = \lim\limits_{t\to 0} f(t,0) = \lim\limits_{t\to 0}\left(-\frac{0}{t^2}\right) = 0$.

(b) Similarly, do it as exercise.

(c) The line $y = x$ has parametric equations $x = t,\ y = mt$, with (0, 0) corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=mx)}} f(x,y) = \lim_{t\to 0} f(t,mt) = \lim_{t\to 0}\left(-\frac{m\cancel{t^2}}{(1+m^2)\cancel{t^2}}\right) = \lim_{t\to 0}\frac{-m}{1+m^2} = \frac{-m}{1+m^2}$$

i.e. the limit along $y = mx$ depends upon m. Consequently, it will be different along different lines through origin.

(d) The parabola $y = x^2$ has parametric equations $x = t,\ y = t^2$ with (0, 0) corresponding to $t = 0$, so

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x^2)}} f(x,y) = \lim_{t\to 0} f(t,t^2) = \lim_{t\to 0}\left(-\frac{t^3}{t^2+t^4}\right) = \lim_{t\to 0}\left(-\frac{t}{1+t^2}\right) = 0$$

Two-Paths Test of Limits:

Note that $\lim\limits_{(x,y)\to(x_0 y_0)} f(x,y)$ exist only when it is same along all the possible path along which (x,y) approaches to (x_0, y_0).

Therefore, if we can find out two different smooth curve (path) along which (x,y) approaches (x_0, y_0) but have different value of limit along them, we say function $f(x,y)$ do not have limit when (x,y) approaches (x_0, y_0).

Example

Find $\lim\limits_{(x,y)\to(0,0)} -\frac{xy}{x^2+y^2}$

Soln. Along curve $x = 0$

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } x=0)}} -\frac{xy}{x^2+y^2} = 0$$

Along curve $y = x$,

$$\lim_{\substack{(x,y)\to(0,0)\\ (\text{along } y=x)}} -\frac{xy}{x^2+y^2} = \lim_{x\to 0} -\frac{x^2}{x^2+x^2} = \frac{1}{2}$$

Since, along two different curves, this limit has differential values, the limit does not exist.

Example

Let

$$f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4+y^2}; & \text{if } x^4+y^2 \neq 0 \\ 0; & \text{if } x = y = 0 \end{cases}$$

show that $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Soln. If we approach origin along any axis, $f(x,y) \to 0$. If we approach (0, 0) along line $y = mx$, then $f(x,y) = f(x, mx) = \frac{mx^3}{x^4 + m^2x^2} = \frac{mx}{x^2+m^2} \to 0$ as $x \to 0$. But if we approach (0, 0) along curve $y = mx^2$, then $f(x,y) = f(x, mx^2) = \frac{mx^4}{x^4 + m^2x^4} \to \frac{m}{1+m^2}$ which is different for different values of *m*. Hence, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Example

Show that $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$ does not exist.

Soln. Along $x = my^2$, $\lim_{y\to 0} \frac{2my^4}{y^4(m^2+1)} = \frac{2m}{m^2+1}$ which is different for different values of *m*. Hence, $\lim_{(x,y)\to(0,0)} \frac{2xy^2}{x^2+y^4}$ does not exist.

General Definition of Limit:

Given a function $f : \mathbb{R}^2 \to \mathbb{R}$. We say that $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$, whenever

(i) Every neighbourhood of the point (x_0, y_0) contains a point of the domain of *f* different from (x_0, y_0) and

(ii) For every $\in > 0$, there exists $\delta > 0$ such that if (x, y) is in the domain and satisfies $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$ (circular disk) then $|f(x,y) - L| < \in$

Remark: (1) By a neighbourhood of a point (x_0, y_0) we mean

(A) An open disc of centred at a point (x_0, y_0) of radius *r*, that is

$$D_r(x_0, y_0) = \left\{(x,y) \in \mathbb{R}^2 : \sqrt{(x-x_0)^2 + (y-y_0)^2} < r\right\}$$

OR

(B) A rectangular disc as $D_r(x_0, y_0) = \{(x,y) \in \mathbb{R}^2 : |x - x_0| + |y - y_0| < r\}$

(in definition we may have either circular disc or rectangular disc)

(2) The condition (i) is included because we donot want to consider limits for isolated points of the domain as in that case there is no “limiting process”. and the condition (ii) implies that as the distance between (x, y) and (x_0, y_0) tends to zero, the distance between $f(x, y)$ and *L* tends to zero.

Example: Show that

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0$$

Let $x = r\cos\theta, y = r\sin\theta,$ then we have

$$\left|f(x,y)-0\right| = \left|\frac{xy}{\sqrt{x^2+y^2}}-0\right| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \left|\frac{r^2\cos\theta\sin\theta}{r}\right| = \left|r\cos\theta\sin\theta\right| \le r$$

$$= \sqrt{x^2+y^2} < \in$$

if $x^2 < \frac{\in^2}{2}, y^2 < \frac{\in^2}{2}$ OR if $|x| < \frac{\in}{\sqrt{2}}, |y| < \frac{\in}{\sqrt{2}}$ let $\in = \delta$, then

$$\left|\frac{xy}{\sqrt{x^2+y^2}}-0\right| < \in \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = 0$$

Example: (2) $\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{x^2+y^2} = 0$

Soln. Let $x = r\cos\theta, y = r\sin\theta$, then

$$\left|f(x,y)-0\right| = \left|xy\frac{x^2-y^2}{x^2+y^2}-0\right| = \left|r^2\sin\theta\cos\theta\cdot\frac{r^2\cos 2\theta}{r^2}\right|$$

$$= \left|r^2\sin\theta\cos\theta\cos 2\theta\right| = \left|\frac{r^2}{4}\sin 4\theta\right| \le \frac{r^2}{4}$$

$$\le \frac{r^2}{4} = \frac{x^2+y^2}{4} < \in \text{ if } \sqrt{x^2+y^2} < 4\sqrt{\in} = \delta(\text{let})$$

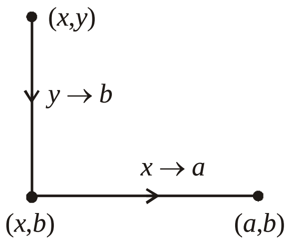
So for every $\in > 0, \exists\, \delta = 2\sqrt{\in} > 0$ such that $\left|f(x,y)-0\right| < \in$ whenever $0 < \sqrt{x^2+y^2} < \delta$

hence $\lim_{(x,y)\to(0,0)} f(x,y) = 0$

Repeated Limits:

We discussed about limits where $(x,y)\to(x_0,y_0)$ along some path *x* and *y* tends to x_0 and y_0 respectively.
Now we discuss the concept of repeated limits.
The repeated limit can be written as

$$\lim_{x\to a}\lim_{y\to b} f(x,y) = \lambda_1$$



If $\lim_{y\to b} f(x,y)$ exist, then it is a function of *x*, say $\phi(x)$. if then the limit, $\lim_{x\to a}\phi(x)$ exists and is equal to λ_1. Then λ is a repeated limit of *f* when first $y\to b$ and then $x\to a$.

Similarly $\lim_{y\to b}\lim_{x\to a} f(x,y) = \lambda_2$

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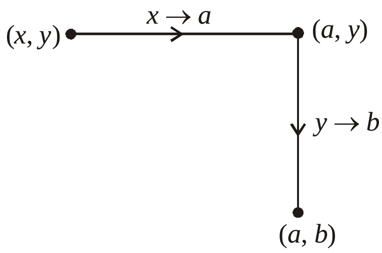

If $\lim\limits_{x\to a} f(x,y)$ exist, then it is a function of y, say $\phi(y)$. If the limit $\lim\limits_{y\to b}\phi(y)$ exists and is equal to λ_2.

Then λ_2 is repeated limit of f when first $x \to a$ and then $y \to b$.

Note that repeated limits may or may not be equal.

Example: (1) Find repeated limits of the function

$$f(x,y)=\begin{cases}\dfrac{2xy}{x^2+y^2}; & (x,y)\neq(0,0)\\ 0 \quad ; & (x,y)=(0,0)\end{cases}$$

Soln. $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} f(x,y)\right)=\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0}\dfrac{2xy}{x^2+y^2}\right)=\lim\limits_{x\to 0}(0)=0$

$\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} f(x,y)\right)=\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0}\dfrac{2xy}{x^2+y^2}\right)=\lim\limits_{y\to 0}(0)=0$

Here both repeated limit exists and are equal.

Example: (2) Find repeated limits of the function

$$f(x,y)=\begin{cases}\dfrac{x-y}{x+y} \quad ; & \text{if } x+y=0\\ 0 \quad ; & \text{if } x+y\neq 0\end{cases}$$

Soln. $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} f(x,y)\right)=\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0}\dfrac{x-y}{x+y}\right)=\lim\limits_{x\to 0}\left(\dfrac{x}{x}\right)=\lim\limits_{x\to 0}1=1$

$\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0} f(x,y)\right)=\lim\limits_{y\to 0}\left(\lim\limits_{x\to 0}\dfrac{x-y}{x+y}\right)=\lim\limits_{y\to 0}\left(\dfrac{-y}{y}\right)=\lim\limits_{y\to 0}(-1)=-1$

Here both repeated limit exists and are not equal.

Remark:

(1) If the repeated limits are not equal, the simultaneous limit cannot exist.

(2) If the simultaneous limit exists then the repeated limits if they exist are necessarily equal but the convense need not be true.

Example: (3) Find $\lim\limits_{(x,y)\to(0,0)}\dfrac{2xy}{x^2+y^2}$

Soln. Let $y=mx$, then we have

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,x)\to(0,0)}\frac{2xy}{x^2+y^2}=\lim_{x\to 0}\frac{2x\cdot mx}{x^2+m^2x^2}=\lim_{x\to 0}\frac{2mx^2}{\left(1+m^2\right)x^2}=\frac{2m}{1+m^2}$$

depends on the choice of slope *m*.

Hence limit do not exist.

Note that for $f(x,y)$ both reapeted limit exists at (0, 0) and are equal.

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Example: (4) Show that limit of the function

$$f(x,y)=\begin{cases} x\sin\left(\frac{1}{y}\right)+y\sin\left(\frac{1}{x}\right); & xy\neq 0 \\ 0 \quad ; & xy=0 \end{cases}$$

at (0, 0) is 0.

Soln. Consider

$$\left|f(x,y)-L\right|=\left|\left(x\sin\left(\frac{1}{y}\right)+y\sin\left(\frac{1}{x}\right)\right)-0\right|=\left|x\sin\left(\frac{1}{y}\right)+y\sin\left(\frac{1}{x}\right)\right|$$

$$\leq\left|x\sin\left(\frac{1}{y}\right)\right|+\left|y\sin\left(\frac{1}{x}\right)\right|<|x|+|y|$$

Let $\in>0$ be given and consider $\in=\delta$, we have $\left|f(x,y)-L\right|<\in$ whenever $0<|x-0|+|y-0|<\delta$

$\Rightarrow \lim\limits_{(x,y)\to(0,0)} f(x,y)=0$

Note: Here both repeated limits do not exist.

as $\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} f(x,y)\right)=\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0}\left(x\sin\left(\frac{1}{y}\right)+y\sin\left(\frac{1}{x}\right)\right)\right)=\lim\limits_{x\to 0}\left(\lim\limits_{y\to 0} x\sin\left(\frac{1}{y}\right)\right)$

as $\lim\limits_{y\to 0}\sin\left(\frac{1}{y}\right)$ do not exist, hence repeated limits do not exist.

Techniques of Finding Limits:

(1) Substitution Method: Sometimes there exist some suitable substitution, which convert limit of function of two variable into limit of function of one variable.

Here note that if we substitute $z=\phi(x,y)$, then $\phi(x,y)$ must be a continuous function:

Example: (1) $\lim\limits_{(x,y)\to(2,-2)}\frac{\sqrt{x-y}-2}{(x-y)-4}$ is

(a) 0 (b) $\frac{1}{4}$ (c) $\frac{1}{3}$ (d) $\frac{1}{2}$

Soln. Let $z=x-y$, then as $(x,y)\to(2,-2)$ we have $z\to 2-(-2)=4$

$$\therefore \lim_{(x,y)\to(2,-2)}\frac{\sqrt{x-y}-2}{(x-y)-4}=\lim_{z\to 4}\frac{\sqrt{z}-2}{z-4}$$

$$=\lim_{z\to 4}\frac{\left(\sqrt{z}-2\right)}{\left(\sqrt{z}+2\right)\left(\sqrt{z}-2\right)}=\lim_{z\to 4}\frac{1}{\sqrt{z}+2}=\frac{1}{\sqrt{4}+2}=\frac{1}{4}$$

Example: (2) Find = $\lim\limits_{(x,y)\to(0,0)}\frac{\sin\left(x^2+y^2\right)}{x^2+y^2}$

Soln. If we let $z = x^2 + y^2$, then

as $(x, y) \to (0,0)$ we have $z \to (0)^2 + (0)^2 = 0$

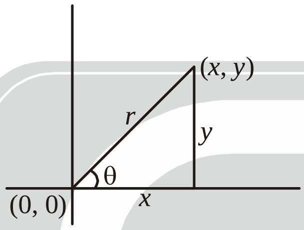
$$\therefore \lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{(x^2+y^2)} = \lim_{z\to 0} \frac{\sin(z)}{z} = 1$$

(2) Using Polar Co-ordinate:

Let $x = r\cos\theta,\ y = r\sin\theta$ then we have

$$r = \sqrt{x^2+y^2} \text{ and } \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Geometrically, r represents the distance between the point (x, y) and origin (0, 0), and θ denote the direction (angle) of point (x, y) with respect to positive X-axis.



Example: (1) $f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2}; & (x,y) \neq (0,0) \\ 0 \quad ; & (x,y) = (0,0) \end{cases}$

Soln. $\therefore$ let $x = r\cos\theta,\ y = r\sin\theta$, then $(x, y) \to (0,0)$ implies that $r \to 0$

we have

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = \lim_{r\to 0}\left(\frac{r^3\cos^3\theta}{r^2}\right) = \lim_{r\to 0} r\cos^3\theta$$

Note that $0 \le |f(r,\theta)| = |r\cos^3\theta| \le r \to 0$ as $r \to 0$ hence by squeeze theorem

$$\lim_{r\to 0} f(r,\theta) = \lim_{r\to 0} r\cos^3\theta = 0$$

$\therefore$ limit exists and equal to zero.

Example: (2) $f(x,y) = \begin{cases} \dfrac{2x^2y}{x^4+y^2}; & (x,y) \neq (0,0) \\ 0 \quad ; & (x,y) = (0,0) \end{cases}$

Soln. Let $x = r\cos\theta,\ y = r\sin\theta$, we get

$$f(r,\theta) = \frac{2r^2\cos^2\theta \cdot r\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta} = \frac{2r^3\cos^2\theta\sin\theta}{r^4\cos^4\theta + r^2\sin^2\theta}$$

$$= \frac{2r\cos^2\theta\sin\theta}{r^2\cos^4\theta+\sin^2\theta}$$

For any $r > 0$, the denominator is > 0.

Since $\left|\cos^2\theta\sin\theta\right| \le 1$, so we tend to think for a while that this limit goes to zero as $r \to 0$.

But if we take the path $r\sin\theta = r^2\cos^2\theta$ we have

$$f(r,\theta) = \frac{2\sin^2\theta}{2\sin^2\theta} = 1\ \forall\,\theta$$

hence limit of $f(x,y)$ does not exist as $(x,y)\to(0,0)$.

Remark: Geometrically *r* denote the distance of point (x, y) to origin and θ denote the direction.
In a bove example, if we consider θ as constant, then

$$\lim_{r\to 0}\frac{2r\cos^2\theta\sin\theta}{r^2\cos^4\theta+\sin^2\theta} = 0$$

But θ is not constant as $\theta\in[0, 2\pi]$ be arbitrary.
As if for some θ denominator approches to zero or in other words if the function of θ is not bounded for $\theta\in(0, 2\pi)$, then we say that limit do not exist.

We can say that for existence of limit function of θ should be bounded for all values of θ in $[0, 2\pi]$.

Algebra of Limit: If *f* and *g* are two functions defined on some neighbourhood of a point (x_0, y_0) such that $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y) = \ell$ and $\lim\limits_{(x,y)\to(x_0,y_0)} g(x,y) = m$ then

(1) $\lim\limits_{(x,y)\to(x_0,y_0)} (f \pm g) = \lim\limits_{(x,y)\to(x_0,y_0)} f \pm \lim\limits_{(x,y)\to(x_0,y_0)} g = \ell \pm m$

(2) $\lim\limits_{(x,y)\to(x_0,y_0)} (f \cdot g) = \lim\limits_{(x,y)\to(x_0,y_0)} f \cdot \lim\limits_{(x,y)\to(x_0,y_0)} g = \ell \cdot m$

(3) If $g(x,y)\neq 0$ for all (x,y) and $m\neq 0$, then

$$\lim_{(x,y)\to(x_0,y_0)}\frac{f}{g}(x,y) = \frac{\lim\limits_{(x,y)\to(x_0,y_0)} f}{\lim\limits_{(x,y)\to(x_0,y_0)} g} = \frac{\ell}{m}$$

Example

Show that $\lim\limits_{(x,y)\to(0,0)}\dfrac{x\sin(x^2+y^2)}{x^2+y^2} = 0$

Soln. $\lim\limits_{(x,y)\to(0,0)}\dfrac{x\sin(x^2+y^2)}{x^2+y^2} = \lim\limits_{(x,y)\to(0,0)} x \lim\limits_{(x,y)\to(0,0)}\dfrac{\sin(x^2+y^2)}{(x^2+y^2)} = 0\times 1 = 0$

Continuity:

Let f be a real valued function defined in a ball around (x_0, y_0). Then

f is said to be continuous at a point (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

Definition: A function $f(x,y)$ is said to be continuous at a point (x_0,y_0) of its domain of definition if for any $\in > 0,\ \exists\ \delta > 0$ such that $\left|f(x,y) - f(x_0,y_0)\right| < \in$ whenever $0 < |x - x_0| + |y - y_0| < \delta$ or $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

Example

Show that the function

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}; & (x,y) \neq (0,0) \\ 0\ ; & (x,y) = (0,0) \end{cases}$$

is continuous at the origin.

Soln. Let $x = r\cos\theta,\ y = r\sin\theta$

$$\therefore \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = r\left|\cos\theta\sin\theta\right| \le r = \sqrt{x^2+y^2} < \varepsilon$$

if $x^2 < \dfrac{\varepsilon^2}{2},\quad y^2 < \dfrac{\varepsilon^2}{2}$ or if $|x| < \dfrac{\varepsilon}{\sqrt{2}},\ |y| < \dfrac{\varepsilon}{\sqrt{2}}$

Thus $\left|\dfrac{xy}{\sqrt{x^2+y^2}} - 0\right| < \varepsilon$, whenever $|x| < \dfrac{\varepsilon}{\sqrt{2}},\ |y| < \dfrac{\varepsilon}{\sqrt{2}}$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0 = f(0,0)$$

Hence, $f(x,y)$ is continuous at origin.

Example

Evaluate $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\,\ell n\,(x^2+y^2)$

Soln. Changing to polar coordinates, $x = r\cos\theta,\quad y = r\sin\theta,\quad r^2 = x^2+y^2$

$$\lim_{(x,y)\to(0,0)} (x^2+y^2)\,\ell n\,(x^2+y^2) = \lim_{r\to 0^+} r^2\,\ell n\, r^2 = \lim_{r\to 0^+} \frac{2\,\ell n\, r}{1/r^2} = \lim_{r\to 0^+} \frac{2/r}{-2/r^3} = \lim_{r\to 0^+} (-r^2) = 0$$

Thus the function $(x^2+y^2)\,\ell n\,(x^2+y^2)$ has a removable discontinuity at (0, 0).
i.e.,

$$f(x,y) = \begin{cases} (x^2+y^2)\,\ell n\,(x^2+y^2); & (x,y) \neq (0,0) \\ 0; & (x,y) = (0,0) \end{cases}$$

is continuous at (0, 0).

Example

Show that

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right) & ;\ (x,y) \neq (0,0) \\ 0 & ;\ (x,y) = (0,0) \end{cases}$$

is continuous at (0, 0).

Soln. To determine the limit, $\lim_{(x,y)\to(0,0)} f(x, y)$, let $x^2+y^2=r^2$ and let $r^2\to 0$. Because,

$$\left|\sin\frac{1}{(x^2+y^2)}\right|\le 1,\ \lim_{(x,y)\to(0,0)}(x^2+y^2)\sin\frac{1}{(x^2+y^2)}=0.$$

independent of the manner in which (x, y) approaches (0, 0). The condition of continuity is satisfied, so $f(x, y)$ is continuous at (0, 0).

Note: (1) If a function is not continuous at (x_0, y_0), we say that f is discontinuous at (x_0, y_0).

Also we classified discontinuity at a point as removable or non-removable discontinuity. Depending on the fact that whether $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exist or not.

If $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists but is not equal to $f(x_0,y_0)$ then we say discontinuity at (x_0,y_0) is removable.

Which can be removed by defining new function as

$$g(x,y)=\begin{cases} f(x,y); & (x,y)\ne(x_0,y_0)\\ L\qquad ; & (x,y)=(x_0,y_0)\end{cases}$$

where L is limit of function at (x_0,y_0).

If $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exists, we say that function has a non-removable discontinuity at (x_0,y_0).

Note: (2) In the calculus of one variable, we studied the properties of continuous functions. Two of most important properties are

(1) Image of a compact set under a continuous map is compact.

(2) Image of a connected st under a continuous map is connected.

In the case of function of two variables also continuous function preserves both the properties.

PRACTICE SET

1. Evaluate $\lim_{(x,y)\to(0,0)}\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}$.

2. Evaluate $\lim_{(x,y)\to(0,0)}\left(\frac{xy}{\sqrt{x^2+y^2}}\right)$.

3. Find the limits of the following if exists

(a) $\lim_{(x,y)\to(0,0)}\frac{xy}{x^2+y^2}$ (b) $\lim_{(x,y)\to(0,0)}\frac{x+\sqrt{y}}{x^2+y^2}$ (c) $\lim_{(x,y)\to(0,0)}\frac{x^3y}{x^6+y^2}$ (d) $\lim_{(x,y)\to(0,1)}\tan^{-1}\left(\frac{y}{x}\right)$

4. Check whether the following functions are continuous or not at the point (0, 0).

(a) $$f(x,y)=\begin{cases}\dfrac{2x^4+3y^4}{x^2+y^2} & ;\ (x,y)\ne(0,0)\\ 0 & ;\ (x,y)=(0,0)\end{cases}$$

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(b) $f(x,y)=\begin{cases}\dfrac{2x(x^2-y^2)}{x^2+y^2} & ; (x,y)\neq(0,0)\\ 0 & ; (x,y)=(0,0)\end{cases}$

(c) $f(x,y)=\begin{cases}\dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} & ; (x,y)\neq(0,0)\\ 1/2 & ; (x,y)=(0,0)\end{cases}$

5. Check whether the following functions are continuous or not.

(a) $f(x,y)=\begin{cases}\dfrac{x-y}{x+y} & ; (x,y)\neq(0,0)\\ 0 & ; (x,y)=(0,0)\end{cases}$ at the point $(0,0)$.

(b) $f(x,y)=\begin{cases}\dfrac{x^2-x\sqrt{y}}{x^2+y} & ; (x,y)\neq(0,0)\\ 0 & ; (x,y)=(0,0)\end{cases}$ at the point $(0,0)$.

(c) $f(x,y)=\begin{cases}\dfrac{x^2+xy+x+y}{x+y} & ; (x,y)\neq(2,2)\\ 4 & ; (x,y)=(2,2)\end{cases}$ at the point $(2,2)$.

6. Let $f(x,y)=\begin{cases}\dfrac{x^4y-3x^2y^2+y^5}{(x^2+y^2)^2} & ; (x,y)\neq(0,0)\\ 0 & ; (x,y)=(0,0)\end{cases}$. Find a $\delta>0$ such that $|f(x,y)-f(0,0)|<0.01$, whenever $\sqrt{x^2+y^2}<\delta$.

SOLUTIONS

1. $\left|\dfrac{\sqrt{x^2y^2+1}-1}{x^2+y^2}\right|=\left|\dfrac{(1+x^2y^2)^{1/2}-1}{x^2+y^2}\right|\le\dfrac{\frac{1}{2}x^2y^2}{x^2+y^2}<\varepsilon$

$\Leftrightarrow \dfrac{1}{2\left(\dfrac{1}{x^2}+\dfrac{1}{y^2}\right)}<\varepsilon \quad\Leftrightarrow\quad 2\left(\dfrac{1}{x^2}+\dfrac{1}{y^2}\right)>\dfrac{1}{\varepsilon}\quad\Leftrightarrow\quad \dfrac{1}{x^2}+\dfrac{1}{y^2}>\dfrac{1}{2\varepsilon}$

if $\dfrac{1}{x^2}>\dfrac{1}{4\varepsilon};\dfrac{1}{y^2}>\dfrac{1}{4\varepsilon}$ or if $x^2<4\varepsilon\,;\,y^2<4\varepsilon$ or if $|x|<2\sqrt{\varepsilon}=\delta\,;\,|y|<2\sqrt{\varepsilon}=\delta$

Thus for any $\varepsilon>0,\exists\,\delta>0$ such that $\left|\dfrac{\sqrt{x^2y^2+1}-1}{x^2+y^2}-0\right|<\varepsilon$ whenever $|x|<\delta,|y|<\delta$.

$$\Rightarrow \boxed{\lim_{(x,y)\to(0,0)}\frac{\sqrt{x^2y^2+1}-1}{x^2+y^2}=0}$$

2. Here $f(x, y) = \frac{xy}{\left(\sqrt{x^2+y^2}\right)}$ is not defined at (0, 0). We have

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \le \frac{1}{2}\frac{(x^2+y^2)}{\sqrt{x^2+y^2}} = \frac{1}{2}\sqrt{x^2+y^2} < \varepsilon, (x, y) \ne (0,0)$$

Since $|xy| \le (x^2+y^2)/2$. If we choose $\delta < 2\varepsilon$, then we get

$$\left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right| < \varepsilon, \text{ whenever } 0 < \sqrt{x^2+y^2} < \delta.$$

Hence, $\lim\limits_{(x, y)\to(0, 0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$

Alternative Writing $x = r\cos\theta, y = r\sin\theta$, we obtain

$$\lim_{(x, y)\to(0, 0)} \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \lim_{r\to 0}\left|\frac{r^2\sin\theta\cos\theta}{r}\right| = 0 \text{ which is independent of } \theta.$$

3. The limit does not exist if it is not finite, or if it depends on a particular path.

(a) Consider the path $y = mx$. As $(x, y) \to (0,0)$, we get $x \to 0$.

Therefore, $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim\limits_{x\to 0} \frac{mx^2}{(1+m^2)x^2} = \frac{m}{1+m^2}$ which depends on m. For different values of m, we obtain different limits. Hence, the limit does not exist.

Alternative Setting $x = r\cos\theta, y = r\sin\theta$, we obtain

$\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2} = \lim\limits_{r\to 0} \frac{r^2\sin\theta\cos\theta}{r^2} = \sin\theta\cos\theta$ which depends on θ. Hence, the limit is dependent on different radial paths $\theta =$ constant. Hence, the limit does not exist.

(b) Choose the path $y = mx^2$. As $(x, y) \to (0,0)$, we get $x \to 0$.

Therefore, $\lim\limits_{(x,y)\to(0,0)} \frac{x+\sqrt{y}}{x^2+y} = \lim\limits_{x\to 0} \frac{1+\sqrt{m}}{(1+m)x} = \infty$.

Since the limit is not finite, the limit does not exist.

(c) Choose the path $y = mx^3$. As $(x, y) \to (0,0)$, we get $x \to 0$.

Therefore, $\lim\limits_{(x,y)\to(0,0)} \frac{x^3y}{x^6+y^2} = \lim\limits_{x\to 0} \frac{mx^6}{(1+m^2)x^6} = \frac{m}{1+m^2}$.

which depends on m. For different values of m, we obtain different limits. Hence, the limit does not exist.

(d) Consider $\lim\limits_{\substack{y\to 1\\ x\to 0^-}} \tan^{-1}\frac{y}{x} = \frac{-\pi}{2}, \lim\limits_{\substack{y\to 1\\ x\to 0^+}} \tan^{-1}\frac{y}{x} = \frac{\pi}{2}$

Since on these two paths limits are different,

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$\therefore \lim_{(x,y)\to(0,1)} \tan^{-1}\frac{y}{x}$ does not exist

4. (a) Let $x = r\cos\theta,\ y = r\sin\theta$. Then, $r = \sqrt{x^2+y^2} \neq 0$. We have

$$|f(x,y) - f(0,0)| = \left|\frac{2x^4+3y^4}{x^2+y^2}\right| = \left|\frac{r^4(2\cos^4\theta + 3\sin^4\theta)}{r^2(\cos^2\theta+\sin^2\theta)}\right| < r^2\left[2|\cos^4\theta| + 3|\sin^4\theta|\right] < 5r^2 < \varepsilon$$

or $r = \sqrt{x^2+y^2} < \sqrt{\varepsilon/5}.$

If we choose $\delta < \sqrt{\varepsilon/5}$, we find that $|f(x,y) - f(0,0)| < \varepsilon$, whenever $0 < \sqrt{x^2+y^2} < \delta$.

Therefore, $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$. Hence, $f(x,y)$ is continuous at (0, 0).

(b) Let $x = r\cos\theta,\ y = r\sin\theta$. Then, $r = \sqrt{x^2+y^2} \neq 0$. We have

$$|f(x,y) - f(0,0)| = \left|\frac{2x(x^2-y^2)}{x^2+y^2}\right| = \left|\frac{2r^3(\cos^2\theta - \sin^2\theta)\cos\theta}{r^2(\cos^2\theta+\sin^2\theta)}\right| = |2r\cos 2\theta\cos\theta| \leq 2r < \varepsilon$$

or $r = \sqrt{x^2+y^2} < \varepsilon/2.$

If we choose $\delta < \varepsilon/2$, we find that $|f(x,y) - f(0,0)| < \varepsilon$, whenever $0 < \sqrt{x^2+y^2} < \delta$.

Therefore, $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = 0$. Hence, $f(x,y)$ is continuous at (0, 0).

(c) Let $x + 2y = t$. Therefore, $t \to 0$ as $(x,y) \to (0,0)$.

We can now write $\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{t\to 0}\frac{\sin^{-1}t}{\tan^{-1}2t} = \lim_{t\to 0}\left[\frac{(\sin^{-1}t)/t}{(\tan^{-1}(2t))/(2t)}\right]\left[\frac{t}{2t}\right] = \frac{1}{2}.$

Since $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = \frac{1}{2}$, the given function is continuous at $(x,y) = (0,0)$.

5. (a) Choose the path $y = mx$. As $(x,y) \to (0,0)$, we get $x \to 0$.

Therefore, $\lim_{(x,y)\to(0,0)} \frac{x-y}{x+y} = \lim_{x\to 0}\frac{(1-m)x}{(1+m)x} = \frac{1-m}{1+m}$ which depends on *m*. Since, the limit does not exist, the function is not continuous at (0, 0).

(b) Choose the path $y = m^2x^2$. As $(x,y) \to (0,0)$, we get $x \to 0$.

Therefore, $\lim_{(x,y)\to(0,0)} \frac{x^2 - x\sqrt{y}}{x^2+y} = \lim_{x\to 0}\frac{(1-m)x^2}{(1+m^2)x^2} = \frac{1-m}{1+m^2}$ which depends on *m*. Since the limit does not exist, the function is not continuous at (0, 0).

(c) $\lim_{(x,y)\to(2,2)} f(x,y) = \lim_{(x,y)\to(2,2)} \frac{(x+y)(x+1)}{(x+y)} = \lim_{(x,y)\to(2,2)} (x+1) = 3.$

Since $\lim_{(x,y)\to(2,2)} f(x,y) = f(2,2)$, the function is not continuous at (2, 2).

Note that the point (2, 2) is a point of removable discontinuity.

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6. We have

$$\left|f(x,y)-f(0,0)\right|=\left|\frac{r^5(\cos^4\theta\sin\theta-3\cos^2\theta\sin^3\theta+\sin^5\theta}{r^4(\cos^2\theta+\sin^2\theta)}\right|$$

$$=\left|r(\cos^4\theta\sin\theta-3\cos^2\theta\sin^3\theta+\sin^5\theta\right|\le r(1+3+1)=5r=5\sqrt{x^2+y^2}<0.01.$$

Therefore, $\sqrt{x^2+y^2}\le 0.01/5=0.002$. Hence, $\delta<0.002$.

Solved Examples

1. Let the function $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(0,0)=0$ and $f(x,y)=\dfrac{x^3-y^3}{x^2+y^2}$ for $(x,y)\neq(0,0)$. Then

(a) f is continuous on $\mathbb{R}^2$ **[D.U. 2014]**

(b) f is continuous at all points of $\mathbb{R}^2$ except at (0, 0)

(c) $f_x(0, 0) = f_y(0, 0)$

(d) f is bounded

Soln. $$f(x,y)=\begin{cases}\dfrac{x^3-y^3}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

For continuity at (0, 0)

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{x^3-y^3}{x^2+y^2}$$

Let $x=r\cos\theta,\ y=r\sin\theta$

then $(x,y)\to(0,0)$ implies $r\to 0,\ \ 0\le\theta\le 2\pi$

$$=\lim_{r\to 0}\frac{r^3\cos^3\theta-r^3\sin^3\theta}{r^2}$$

$$=\lim_{r\to 0} r(\cos^3\theta-\sin^3\theta)$$

Since $f(\theta)=\cos^3\theta-\sin^3\theta$ is bounded for $0\le\theta\le 2\pi$

Then by Squeeze theorem

$$=\lim_{r\to 0} r(\cos^3\theta-\sin^3\theta)=0$$

$$\therefore \lim_{(x,y)\to(0,0)} f(x,y)=f(0,0)$$

Partial derivatives

$$f_x(0,0)=\lim_{h\to 0}\frac{f(0+h,0)-f(0,0)}{h}=\lim_{h\to 0}\frac{1}{h}\cdot\frac{h^3}{h^2}=1$$

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$$f_y(0,0)=\lim_{k\to 0}\frac{f(0,0+k)-f(0,0)}{k}=\lim_{k\to 0}\frac{1}{k}\cdot\frac{-k^3}{k^2}=-1$$

Therefore, $f_x(0,0)\neq f_y(0,0)$

Also f is not bounded over $\mathbb{R}^2$, as for large r, $f(x,y)\to+\infty$.

Hence $f(x,y)$ is not bounded by any real number.

Correct option is (a)

2. Determine whether the function **[IIT JAM(MS)-2010]**

$$f(x,y)=\begin{cases}\dfrac{x^4+y^4}{x^2+y^2}, & (x,y)\neq(0,0),\\ 0, & (x,y)=(0,0),\end{cases}$$

is continuous at (0, 0).

Soln. $$f(x,y)=\begin{cases}\dfrac{x^4+y^4}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

For continuity

Let $x=r\cos\theta, y=r\sin\theta$ then $(x,y)\to(0,0)$ implies $r\to 0$

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{r\to 0}\frac{r^4(\cos^4\theta+\sin^4\theta)}{r^2}$$

$$=\lim_{r\to 0} r^2\cdot(\cos^4\theta+\sin^4\theta)=0=f(0,0)$$ (By Squeeze theorem)

hence $f(x,y)$ is continuous at (0, 0)

3. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by

$$f(x,y)=\begin{cases}\dfrac{\sin\left(2\left(x^2+y^2\right)\right)}{x^2+y^2}e^{3x\sin\left(\frac{4}{y}\right)}, & \text{if } (x,y)\neq(0,0),\\ \alpha, & \text{if } (x,y)=(0,0),\end{cases}$$ **[IIT JAM(MS)-2013]**

where α is a real constant. If f is continuous at (0, 0), then α is equal to

(a) 1 (b) 2 (c) 3 (d) 4

Soln. $$f(x,y)=\begin{cases}\dfrac{\sin(2(x^2+y^2))}{x^2+y^2}\cdot e^{3x\sin\left(\frac{4}{y}\right)} & (x,y)\neq(0,0)\\ \alpha & (x,y)=(0,0)\end{cases}$$

If $f(x,y)$ is continuous at (0, 0), then

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$$\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0) = \alpha$$

Now $\lim_{(x,y)\to(0,0)} \dfrac{\sin(2(x^2+y^2))}{x^2+y^2}\cdot e^{3x\sin\left(\frac{4}{y}\right)}$

$$= \lim_{(x,y)\to(0,0)} \frac{\sin 2(x^2+y^2)}{x^2+y^2}\cdot \lim_{(x,y)\to(0,0)} e^{3x\sin\left(\frac{4}{y}\right)}$$

Note that, let $z = x^2+y^2$, then $(x,y)\to(0,0)$ implies $z\to 0$

$$= \lim_{(x,y)\to(0,0)} \frac{\sin 2(x^2+y^2)}{x^2+y^2} = \lim_{z\to 0}\frac{\sin 2z}{2} = \lim_{z\to 0} 2\cdot\frac{\sin 2z}{2z} = 2$$

and $\lim_{(x,y)\to(0,0)} e^{3x\sin\left(\frac{4}{y}\right)} = e^{\lim_{(x,y)\to(0,0)} 3x\sin\left(\frac{4}{y}\right)}$

Since $-3x \le 3x\sin\left(\frac{4}{y}\right) = 3x$ for all (x, y)

and $\lim_{(x,y)\to(0,0)} 3x = 0$, then by Squeeze theorem.

$$\lim_{(x,y)\to(0,0)} 3x\sin\left(\frac{4}{y}\right) = 0$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} e^{3x\sin\left(\frac{4}{y}\right)} = e^0 = 1$$

hence $\lim_{(x,y)\to(0,0)} f(x,y) = 2\cdot 1 = 2$

hence $\alpha = 2$,

Correct option is (b)

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EXERCISE-1

1. Show that

(i) $\lim_{(x,y)\to(0,0)}\left(\frac{1}{|x|}+\frac{1}{|y|}\right)=\infty$

(ii) $\lim_{(x,y)\to(0,0)}\frac{xy^2}{x^2+y^2}=0$

(iii) $\lim_{(x,y)\to(0,0)}(x+y)=0$

(iv) $\lim_{(x,y)\to(0,0)}\left(\frac{1}{xy}\right)\sin\left(x^2y+xy^2\right)=0$

2. Show that the limit, when $(x,y)\to(0,0)$ does not exist in each case.

(i) $\lim\frac{2xy}{x^2+y^2}$

(ii) $\lim\frac{xy^3}{x^2+y^6}$

(iii) $\lim\frac{x^2y^2}{x^2y^2+(x^2-y^2)^2}$

(iv) $\lim\frac{x^3+y^3}{x-y}$

3. Show that the limit, when $(x,y)\to(0,0)$ exist in each case.

(i) $\lim\frac{xy}{\sqrt{x^2+y^2}}$

(ii) $\lim\frac{x^3y^3}{x^2+y^2}$

(iii) $\lim\frac{x^3-y^3}{x^2+y^2}$

(iv) $\lim\frac{x^4+y^4}{x^2+y^2}$

4. Check whether the following functions are continuous or discontinuous at the origin:

(i) $f(x,y)=\begin{cases}\frac{1}{x^2+y^2}; & (x,y)\neq(0,0)\\ 0; & (x,y)=(0,0)\end{cases}$

(ii) $f(x,y)=\begin{cases}\frac{x^4-y^4}{x^4+y^4}; & (x,y)\neq(0,0)\\ 0; & (x,y)=(0,0)\end{cases}$

(iii) $f(x,y)=\begin{cases}\frac{x^2y^2}{x^4+y^4}; & (x,y)\neq(0,0)\\ 0; & (x,y)=(0,0)\end{cases}$

(iv) $f(x,y)=\begin{cases}\frac{x^2y^2}{(x^2+y^2)}; & (x,y)\neq(0,0)\\ 0; & (x,y)=(0,0)\end{cases}$

(v) $f(x,y)=\begin{cases}\frac{x^3y^3}{(x^2+y^2)}; & (x,y)\neq(0,0)\\ 0; & (x,y)=(0,0)\end{cases}$

5. Can the given functions be appropriately defined at (0, 0) in order to be continuous there ?

(i) $f(x,y)=|x|^y$

(ii) $f(x,y)=\sin\frac{x}{y}$

(iii) $f(x,y)=\frac{x^3+y^3}{x^2+y^2}$

(iv) $f(x,y)=x^2\log(x^2+y^2)$

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