$$= a + ar + ar^2 + ... + ar^{n-1} = \begin{cases} \frac{a(1-r^n)}{1-r}, & |r|<1 \\ na, & r=1 \\ \frac{a(r^n-1)}{r-1}, & |r|>1 \\ \frac{1-(-1)^n}{2}, & r=-1 \end{cases}$$

When $| r | < 1$ $\lim_{n\to\infty} S_n = \frac{a}{1-r}$ $\left(\because \lim_{n\to\infty} r^n = 0\right)$

$\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$ is convergent

When $r=1, \lim_{n\to\infty} S_n = \infty \Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$ is divergent

When $|r|>1, \lim_{n\to\infty} S_n = \pm\infty \Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$ is divergent

When $r=-1,\ S_{2n}=0, S_{2n+1}=1 \Rightarrow (S_n)$ is having two subsequence converging to different limits.

$\Rightarrow \lim_{n\to\infty} s_n$ does not exist $\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$ is divergent.

$\therefore$ The geometric series $\sum_{n=1}^{\infty} ar^{n-1}$ converges iff $|r|<1$.

Theorem: Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Then $\lim_{n\to\infty} a_n = 0$.

Remark: The converse of this theorem is not true. For example consider $\sum_{n=1}^{\infty} \frac{1}{n}$.

here $a_n = \frac{1}{n} \forall n$ and $\lim_{n\to\infty} a_n = 0$

However, it can be shown that $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series.

Example: Consider $\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}\right)$. Show that this series is convergent.

Soln: Let $a_n = \frac{1}{\sqrt{n+1}} - \frac{1}{\sqrt{n+2}}$

Then $S_n = a_1 + a_2 + ... + a_n = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{n+2}}$

$\Rightarrow \lim_{n\to\infty} S_n = \frac{1}{\sqrt{2}} \Rightarrow \sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} a_n = \frac{1}{\sqrt{2}}$

Example: Consider $\sum_{n=1}^{\infty}\frac{\underline{|n}}{n^2}$. Show that this series is divergent

Soln: Let $a_n=\frac{\underline{|n}}{n^2}=\frac{n(n-1)\underline{|n-2}}{n^2}=\left(1-\frac{1}{n}\right)\underline{|n-2}\ \forall n\geq 2$

Then $\lim_{n\to\infty}a_n=\lim_{n\to\infty}\left(1-\frac{1}{n}\right)\underline{|n-2}=\infty$

$\because \lim_{n\to\infty}a_n\neq 0\Rightarrow$ By the above theorem $\sum_{n=1}^{\infty}\frac{\underline{|n}}{n^2}$ is divergent

Example: Show that $\sum_{n=1}^{\infty}\frac{2n+1}{3n+2}$ is divergent.

Soln: Let $a_n=\frac{2n+1}{3n+2}$

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\left(\frac{2n+1}{3n+2}\right)=\lim_{n\to\infty}\frac{2+\frac{1}{n}}{3+\frac{2}{n}}=\frac{2}{3}$$

$\because \lim_{n\to\infty}a_n=\frac{2}{3}\neq 0\Rightarrow\sum_{n=1}^{\infty}a_n$ is divergent.

6.2.1.Theorem: (Cauchy's criterion for series):

The series $\sum_{n=1}^{\infty}a_n$ converges if and only if for every $\varepsilon>0$, $\exists N\in\mathbb{N}$ s.t. $|S_m-S_n|=|a_{n+1}+a_{n+2}+...+a_m|<\varepsilon$ for all $m>n\geq N$.

Example: (Harmonic series): Show that the series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Soln. We note that

$$|S_{2n}-S_n|=\left|\sum_{k=n+1}^{2n}\frac{1}{k}\right|=\left|\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}\right|=\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}$$

$$\geq\frac{1}{2n}+\frac{1}{2n}+...+\frac{1}{2n}=n.\frac{1}{2n}=\frac{1}{2} \qquad ...(1)$$

Suppose that $\sum_{n=1}^{\infty}\frac{1}{n}$ converges. By Cauchy's criterion, given $\varepsilon=\frac{1}{2},\exists N\in\mathbb{N}$ s.t. $\left|\sum_{k=n+1}^{m}\frac{1}{k}\right|<\frac{1}{2}$ for all $m>n\geq N$.

If we choose $n=N$, $m=2N$ then

$\left|\sum_{k=N}^{2N}\frac{1}{k}\right|<\frac{1}{2}$ i.e. $|S_{2N}-S_N|<\frac{1}{2}$ which is a contradiction to (1) $\Rightarrow\sum_{n=1}^{\infty}\frac{1}{n}$ is divergent.

6.2.2.Theorem:

Let (a_n) be a sequence of non-negative real numbers. Then $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence of partial sums is bounded.

Example: Show that the exponential series $\sum_{n-0}^{\infty} \frac{1}{\lfloor n}$ is bounded.

Soln. $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{\lfloor n}, \quad a_n = \frac{1}{\lfloor n} > 0$

Using the above theorem, if we can show that (S_n) is bounded, then $\sum_{n=0}^{\infty} a_n$ will be convergent.

Now $S_n = a_0 + a_1 + ... + a_n = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + ... + \frac{1}{\lfloor n}$

Now $\frac{1}{\lfloor n} = \frac{1}{1.2.3...(n-1)n} \le \frac{1}{2^{n-1}}$

$\therefore \; S_n \le 1 + 2\left[1 - \left(\frac{1}{2}\right)^n\right] = 3 - \frac{1}{2^{n-1}}$

$\Rightarrow \; S_n < 3 \; \forall n \in \mathbb{N} \quad \Rightarrow \; (S_n)$ is bounded $\Rightarrow \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{\lfloor n}$ is convergent

6.3. Tests for Convergence of Series

5.3.1 p-Series Test:

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \le 1$.

6.3.2. Comparison Test:

(a) Let (a_n) and (b_n) be two sequences such that $|a_n| \le b_n$ for some $n \ge N_0$, where N_0 is some fixed integer. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(b) Let (d_n) be a sequence such that $a_n \ge d_n \ge 0$ for some $n \ge N_0$. If $\sum_{n=1}^{\infty} d_n$ diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

Example: Examine the convergence of the following series

(a) $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ (b) $\sum_{n=1}^{\infty} \frac{n+1}{n^2+1}$

Soln. (a) Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ i.e. $a_n = \frac{1}{n^2+1} \quad \because \quad \frac{1}{n^2+1} \le \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent $(\because p = 2)$

$\Rightarrow$ By comparison test $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ is convergent