$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2-r^2}} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{(1-0)}{n}\frac{1}{\sqrt{4-\left(0+r\left(\frac{1-0}{n}\right)\right)^2}}$$

Which is of the form $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{b-a}{n}f\left(a+r\left(\frac{b-a}{n}\right)\right)$

Here $b=1,\ a=0$ and $f(x)=\frac{1}{\sqrt{4-x^2}}$

So, $L=\int_0^1\frac{dx}{\sqrt{4-x^2}} = \sin^{-1}\frac{x}{2}\Big|_0^1 = \frac{\pi}{6}.$

Ex.18: Evaluate : $\lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots+\frac{1}{64n}\right]$

Soln. $L=\lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots+\frac{1}{64n}\right] = \lim_{n\to\infty}\sum_{r=1}^{3n}\frac{n^2}{(n+r)^3}$

Put $3n=m$, we get, $L=\lim_{n\to\infty}\sum_{r=1}^{m}\frac{m^2/9}{\left(\frac{m}{3}+r\right)} = \lim_{n\to\infty}\sum_{r=1}^{m}\frac{3}{m}\left(\frac{1}{1+\frac{3r}{m}}\right)^3 = \int_0^3\frac{dx}{(1+x)^3} = \frac{-1}{2(1+x)^2}\Big|_0^3 = \frac{15}{32}.$

5. DEFINITE INTEGRALS DEPENDENT ON PARAMETERS

Let given integral is $I(\alpha)=\int_a^b f(x,\alpha)\,dx,\ a\le x\le b$

Here,
(i) x is variable of integration
(ii) α is a parameter independent of x
(iii) a and b are constants
If this integral can not be solved directly then partially differentiating both sides w.r.t. α

$$\frac{dI}{d\alpha}=\int_a^b\frac{\partial}{\partial\alpha}\left(f(x,\alpha)\right)dx$$

Now, $\frac{\partial}{\partial\alpha}f(x,\alpha)$ is easily integrable $\frac{dI}{d\alpha}=g(\alpha)$

$I=\int g(\alpha)\,d\alpha$; $I=h(\alpha)+c$

c can be found by initial integral.

Ex.19: Evaluate : $\int_0^1\frac{x^b-1}{\log_e x}$, '$b$' being parameter

Soln. Let $I(b)=\int_0^1\frac{x^b-1}{\log_e x}dx = \frac{dI(b)}{db}=\int_0^1\frac{x^b\log_e x}{\log_e x}dx+0-0$ (using modified Leibnitz theorem)

$$=\int_0^1 x^b\,dx=\frac{x^{b+1}}{b+1}\Big]_0^1=\frac{1}{b+1}$$

$I(b) = \log_e (b+1) + c$

$b = 0 \Rightarrow I(0) = 0$

$c = 0 \quad \therefore I(b) = \log_e (b+1)$

Ex.20: Evaluate : $\int_0^1 \frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}} dx$, 'a' being parameter

Soln. Let $I(a) = \int_0^1 \frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}} dx$

$$\frac{dI(a)}{da} = \int_0^1 \frac{x}{(1+a^2x^2)} \frac{1}{x\sqrt{1-x^2}} dx = \int_0^1 \frac{dx}{(1+a^2x^2)\sqrt{1-x^2}}$$

Put $x = \sin t \quad \Rightarrow \quad dx = \cos t \, dt$

L.L.: $x = 0 \quad \Rightarrow \quad t = 0$

U.L.: $x = 1 \quad \Rightarrow \quad t = \frac{\pi}{2}$

$$\frac{dI(a)}{da} = \int_0^{\pi/2} \frac{1}{1+a^2\sin^2 t} \frac{1}{\cos t} \cos t \, dt = \int_0^{\pi/2} \frac{dt}{1+a^2\sin^2 t}$$

$$= \int_0^{\pi/2} \frac{\sec^2 t \, dt}{1+(1+a^2)\tan^2 t} = \frac{1}{\sqrt{1+a^2}} \tan^{-1}\left(\sqrt{1+a^2}\tan t\right)\Big]_0^{\pi/2} = \frac{1}{\sqrt{1+a^2}} \cdot \frac{\pi}{2}$$

$\Rightarrow \quad I(a) = \frac{\pi}{2} \log_e \left(a + \sqrt{1+a^2}\right) + c$

But $\quad I(0) = 0 \Rightarrow c = 0$

$\Rightarrow \quad I(a) = \frac{\pi}{2} \log_e \left(a + \sqrt{1+a^2}\right).$

Ex.21: If a real valued function f is given by $\int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x} + b, \, x > 0$, where $a > 0$ and b are real constants, then $f(4)$ is equal to

(a) 4 (b) 6 (c) 8 (d) 10 **[JAM-CA-2010]**

Soln. $\int_a^x \frac{f(t)}{t^2} dt = 2\sqrt{x} + b$

Taking derivative on both sides w.r.t. x

$\frac{f(x)}{x^2} = \frac{1}{\sqrt{x}} \Rightarrow f(x) = x^{3/2}$. So, $f(4) = 4^{3/2} = 8$.

Hence, correct option is (c).

Ex.22: Let $f(x) = \int_{\sin x}^{\cos x} e^{-t^2} dt$, then $f'\left(\frac{\pi}{4}\right)$ equals

(a) $\sqrt{\frac{1}{e}}$ (b) $-\sqrt{\frac{2}{e}}$ (c) $\sqrt{\frac{2}{e}}$ (d) $-\sqrt{\frac{1}{e}}$ **[JAM-MA-2006]**

Soln. Applying Leibnitz rule

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(t) \, dt = f(v(x)) \frac{dv}{dx} - f(u(x)) \frac{du}{dx}$$

$$f(x)=\int_{\sin x}^{\cos x} e^{-t^2}dt$$

$$f'(x)=e^{-\cos^2 x}(-\sin x)-e^{-\sin^2 x}(\cos x)$$

$$f'\left(\frac{\pi}{4}\right)=e^{-1/2}\cdot\left(-\frac{1}{\sqrt{2}}\right)-e^{-1/2}\cdot\left(\frac{1}{\sqrt{2}}\right)=-\sqrt{\frac{2}{e}}.$$

Hence, correct option is (b).

7. BETA AND GAMMA FUNCTION

7.1 BETA FUNCTION

Beta function is denoted by $B(l, m)$ and defined as $B(l, m)=\int_0^1 x^{l-1}(1-x)^{m-1}dx$, where l, m are positive numbers.

7.1.1 Symmetric property of Beta function $B(l, m)=B(m, l)$

Proof : $B(l, m)=\int_0^1 x^{l-1}(1-x)^{m-1}dx=\int_0^1 (1-x)^{l-1}\{1-(1-x)\}^{m-1}dx=\int_0^1 x^{m-1}(1-x)^{l-1}dx=B(m, l).$

7.2 GAMMA FUNCTION

Gamma function denoted by $\Gamma n=\int_0^\infty e^{-x}x^{n-1}dx$, $n>0$.

7.2.1 RELATION BETWEEN BETA FUNCTION AND GAMMA FUNCTION

$$B(l, m)=\frac{\Gamma l\,\Gamma m}{\Gamma(l+m)}$$

7.2.2 PROPERTIES OF GAMMA FUNCTION

(i) $\Gamma n=(n-1)\,\Gamma(n-1)$
If n is positive integer
$\Gamma n=(n-1)!$

(ii) $\Gamma 1=1$

(iii) $\Gamma 0=\infty$

(iv) $\Gamma(-n)=\infty$ where n is positive integer

(v) $\Gamma\frac{1}{2}=\sqrt{\pi}$

(vi) $\Gamma\frac{1}{n}\,\Gamma\left(1-\frac{1}{n}\right)=\frac{\pi}{\sin\frac{\pi}{n}}$, n is a positive integer.

7.2.3 TRANSFORMATION OF GAMMA FUNCTION

(i) Put $x=\log\left(\frac{1}{y}\right)$

$$\Gamma n=\int_0^1\left(\log\frac{1}{y}\right)^{n-1}dy$$

(ii) Put $x=cy$

$$\int_0^\infty e^{-cy}y^{n-1}dy=\frac{\Gamma n}{c^n}$$