The Gauss-seidel method is given by

$$x_1^{(n+1)} = \frac{1}{2}\left[7 + x_2^{n}\right]$$

$$x_2^{(n+1)} = \frac{1}{2}\left[1 + x_1^{(n+1)} + x_3^{(n)}\right]$$

$$x_3^{(n+1)} = \frac{1}{2}\left[1 + x_2^{(n+1)}\right]$$

Given, $X^{(0)} = \left[x_1^{(0)}, x_2^{(0)}, x_3^{0}\right] = [0, 0, 0]$

$$x_1^{(1)} = \frac{1}{2}\left[7 + x_2^{(0)}\right] = \frac{1}{2}[7+0] = 3.5$$

$$x_2^{(1)} = \frac{1}{2}\left[1 + x_1^{(1)} + x_3^{(0)}\right] = \frac{1}{2}[1+3.5+0] = 2.25$$

$$x_3^{(1)} = \frac{1}{2}\left[1 + x_2^{(1)}\right] = \frac{1}{2}[1+2.25] = 1.625$$

Hence option (b) is correct.

25. Consider the system of equations **[D.U. 2017]**

$$\begin{bmatrix} 1 & -a \\ -a & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

where ‘a’ is a real constant. Then Gauss-seidal method for the solution of the above system converges for

(a) all values of a (b) $|a| < 1$ (c) $|a| > 1$ (d) $a > 2$

Soln. Consider the system $AX = B$. We know that the gauss seidal method converges for $AX = B$ if the matrix A is diagonally dominated.

i.e. $a_{ii} > \sum_{i \neq j} |a_{ij}|$ for each i

$\therefore$ we have $1 > |a|$

Hence option (b) is correct.

26. The total number of arithmetic operations required to find the solution of a system of n linear equation in n unknowns by Gauss elimination method is **[D.U. 2018]**

(a) $\frac{2}{3}n^3 + \frac{1}{2}n^2 - \frac{5}{6}n$ (b) $n^3 - \frac{1}{6}n$

(c) $\frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{7}{6}n$ (d) $\frac{1}{3}n^3 + \frac{1}{2}n^2 - \frac{5}{6}n$

Soln. **Article - Counting operation in Gaussian elimination.**

Consider a system of n linear equation in n unknowns and the corresponding n × n coefficient matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots a_{1n} \\ a_{21} & a_{22} \ldots\ldots a_{2n} \\ | & | \qquad | \\ a_{n1} & a_{n2} \ldots\ldots a_{nn} \end{bmatrix}$$

The $n \times 1$, solution matrix $X = [x_1, x_2 \ldots\ldots x_n]^T$, and the $n \times 1$ constant matrix $B = [b_1, b_2 \ldots\ldots b_n]^T$.

consider $AX = B$

Then After Gaussian elimination, the system convert as

$$UX = G$$

Where $u = \begin{bmatrix} u_{11} & u_{12} \ldots\ldots u_{1n} \\ & u_{22} \ldots\ldots u_{2n} \\ & \vert \\ \ldots\ldots\ldots\ldots & u_{nn} \end{bmatrix}$ is the coefficient matrix of the equivalent system,

and $g = \begin{bmatrix} g_1 \\ g_2 \\ \vert \\ g_n \end{bmatrix}$ be the $n \times 1$ equivalent constant matrix.

The following table states the operation count from going from A to U at each step as

Step Number	Additional subtraction	Number of multiplications	Divisions
1.	$(n-1)^2$	$(n-1)^2$	$n-1$
2.	$(n-2)^2$	$(n-2)^2$	$n-2$
⋮	⋮	⋮	⋮
$n-2$	4	4	2
$n-1$	1	1	1

Therefore

Total number of Addition/Subtraction= $\sum (n-1)^2 = \dfrac{n(n-1)(2n-1)}{6}$

Total number of Multiplication/Divisions $= \sum (n-1)^2 + \sum (n-1) = \dfrac{n(n-1)(2n-1)}{6} + \dfrac{n(n-1)}{2} = \dfrac{n(n^2-1)}{3}$

Now we count the Number of Addition/Subtraction and the Number of Multiplication/divisions in going from $b \to g$, we have

Total Number of Addition/Subtraction $= \sum (n-1) = \dfrac{n(n-1)}{2}$

Total Number of Multiplication/divisions $= \sum (n-1) = \dfrac{n(n-1)}{2}$

Lastly, we count the Number of Addition/Subtraction and Multiplications/divisions for finding the solution from the back substitution method, as

Total Number of addition/subtraction $= \sum (n-1) = \dfrac{n(n-1)}{2}$

Total Number of Multiplication/division $= \sum n = \dfrac{n(n+1)}{2}$

Therefore, the total number of operations to obtain the solution of a system of n linear equations in n variables using Gaussian - elimination method is

Total Number of additions/subtraction $= \dfrac{n(n-1)(2n-1)}{6} + \dfrac{n(n-1)}{2} + \dfrac{n(n-1)}{2} = \dfrac{n(n-1)(2n+5)}{6}$

Total number of multiplication /divisions to obtain solution is

$$=\frac{n(n^2-1)}{3}+\frac{n(n-1)}{2}+\frac{n(n+1)}{2}=\frac{n(n^2+3n+1)}{3}$$

Therefore total number of Arithmetic operations are

$$=\frac{n(n-1)(2n+5)}{6}+\frac{n(n^2+3n-1)}{3}$$

$$=\frac{4n^3+9n^2-7n}{6}=\frac{2}{3}n^3+\frac{3}{2}n^2-\frac{7}{6}n$$

Total number of operation are $=\frac{2}{3}n^3+\frac{3}{2}n^2-\frac{7}{6}n$

Hence option (c) is correct

27. If $\begin{pmatrix}1&4&3\\2&7&9\\5&8&a\end{pmatrix}=\begin{bmatrix}\ell_{11}&0&0\\\ell_{21}&\ell_{22}&0\\\ell_{31}&\ell_{32}&-53\end{bmatrix}\begin{bmatrix}1&u_{12}&u_{13}\\0&1&u_{23}\\0&0&1\end{bmatrix}$, then the value of a is **[GATE-2008]**

(a) –2 (b) –1 (c) 1 (d) 2

Soln. $\begin{pmatrix}1&4&3\\2&7&9\\5&8&a\end{pmatrix}=\begin{bmatrix}\ell_{11}&0&0\\\ell_{21}&\ell_{22}&0\\\ell_{31}&\ell_{32}&-53\end{bmatrix}\begin{bmatrix}1&u_{12}&u_{13}\\0&1&u_{23}\\0&0&1\end{bmatrix}=\begin{bmatrix}\ell_{11}&\ell_{11}u_{12}&\ell_{11}u_{13}\\\ell_{21}&\ell_{11}u_{12}+\ell_{22}&\ell_{21}u_{13}+\ell_{22}u_{23}\\\ell_{31}&\ell_{31}u_{12}+\ell_{32}&\ell_{31}u_{13}+\ell_{32}u_{23}-53\end{bmatrix}$

Comparing both sides, we have

$\ell_{11}=1,\ell_{21}=2,\ell_{31}=5,\ell_{11}u_{12}=4$ implies $u_{12}=4$

$\ell_{11}u_{13}=3$ implies $u_{13}=3,\ell_{21}u_{12}+\ell_{22}=7$ implies $8+\ell_{22}=7$ i.e. $\ell_{22}=-1$

$\ell_{21}u_{13}+\ell_{22}u_{23}=9$ implies $6-u_{23}=9$ i.e. $u_{23}=-3$

$\ell_{31}u_{12}+\ell_{32}=8$ implies $20+\ell_{32}=8$ i.e. $\ell_{32}=-12$

Now, $\ell_{31}u_{13}+\ell_{32}u_{23}-53=a$ implies $15+36-53=a$ i.e. $a=-2$

Option (a) is correct

4

Interpolation by Polynomials

Suppose that a function $f(x)$ is not defined explicitely, but its value at some finite number of points $\{x_i, i=1,2,....,n\}$ is given. The interest is to find the value of f at some point x lying between x_j and x_k, for some $j, k = 1, 2,....,n$. This can be obtained by first approximating f by a known function and then finding the value of this approximate function at the point x. Such a process is called the interpolation.

Lagrange Interpolation:

The basic interpolation problem can be posed in one of two ways:

I. Given a set of nodes $\{x_i / i=0,1,....n\}$ and corresponding date values $\{y_i / i=0,1,....n\}$. Find the polynomial $p_n(x)$ of degree less than or equal to n, such that $p_n(x_i)=y_i$, $i=0,1,...n$

II. Given a set of nodes $\{x_i / i=0,1,....,n\}$ and a continuous function $f(x)$. Find the polynomial $p_n(x)$ of degree less than or equal to n, such that $p_n(x_i)=f(x_i), i=0,1,....n$

Note that in the first problem we are trying to fit a polynomial to the data, and in the second case, we are trying to approximate a given function with the interpolating polynomial. Note that the first problem can be viewed as a particular case of the second.

Theorem (Lagrange Interpolation Formula)

Let $x_0, x_1, ...x_n \in I=[a,b]$ be $n+1$ distinct nodes and let $f(x)$ be a continuous real- valued function defined on I. Then, there exists a unique polynomial p_n of degree $\leq n$ (called Lagrange Formula for interploting Polynomial), given by

$$p_n(x)=\sum_{k=0}^{n} f(x_k) l_k(x) \quad \text{where} \quad l_k(x)=\prod_{i=0, i\neq k}^{n} \frac{x-x_i}{x_k-x_i}, k=0,...,n$$

such that $p_n(x_i)=f(x_i), i=0,1,....,n$

The function $l_k(x)$ is called the **Lagrange Multiplier.**

Newton Interpolation and Divided Differences

We have seen that in the Lagrange formula of interpolating polynomial for a function,if we decide toadd a point to the set of nodes to increase the accuracy, we have to completely recompute all of the $l_i(x)$ functions. In other words, we cannot express P_{n+1} in terms of p_n, using Lagrange formula. An alternate form of the polynomail, known as the Newton form, avoids this problem, and allows us to easily write P_{n+1} in terms of P_n.

The idea behind the Newton formula of the interploting polynomial is to write $P_n(x)$ in the form (called Newton form)