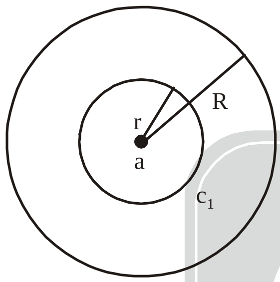
(b) $(z-3)\sin\left(\frac{1}{z+2}\right)=(z-3)\left[\frac{1}{z+2}-\frac{1}{(z+2)^3.3!}+\frac{1}{(z+2)^5.5!}-........\right]$

so, z = -2 is an isolated essential singular point.

5.6 Residue of a Complex Function

Residue at a pole:

Let, z = a be a pole of order 'm' of f (z) and C_1 is a circle of radius 'r' with center at z = a which does not contain singularities except z = a, then f(z) is analytic within the annular region r < | z – a | < R can be expanded into Laurrent series within the annulur region as:



$$f(z)=\sum_{n=0}^{\infty}a_n(z-a)^n+\sum_{n=1}^{\infty}b_n(z-a)^{-n}$$

Co-efficient b_1 is known as residue of f (z) at z = a i.e. Res. f(z = a) = b_1 = $\frac{1}{2\pi i}\oint_{C_1} f(z)dz$

Methods of finding residues:

CASE 1: Residue at simple pole:

(a) Method 1: Res. f(z = a) = $\underset{z\to a}{\text{Lt}}(z-a)f(z)$

(b) Method 2: If f(z) = $\frac{\phi(z)}{\psi(z)}$ where $\psi(a)=0$ but $\phi(a)\neq 0$, then Res. f(z = a) = $\frac{\phi(a)}{\psi'(a)}$

CASE 2: Residue at a pole of order 'n':

(a) Method 1: Res. f (z = a) = $\frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n f(z)\right]\right\}_{z=a}$

(b) Method 2: First put z - a = t and expand it into series, then Res. f (z = a) = co-efficient of 1/t.

CASE 3: Residue at z = ∞:

$$\text{Res.} f(z=\infty)=\underset{z\to\infty}{Lt}\left[-zf(z)\right]$$

Example 30: Find the singular points of the following function and the corresponding residues:

(a) $f(z)=\frac{1-2z}{z(z-1)(z-2)}$ (b) $f(z)=\frac{z^2}{z^2+a^2}$ (c) $f(z)=z^2e^{1/z}$

Soln: (a) $f(z)=\dfrac{1-2z}{z(z-1)(z-2)} \Rightarrow \text{Poles}: z=0, z=1, z=2$

$$\text{Res. } f(z=0)=\underset{z\to 0}{Lt}(z-0)\,f(z)=\underset{z\to 0}{Lt}\frac{1-2z}{(z-1)(z-2)}=\frac{1}{2}$$

$$\text{Res. } f(z=1)=\underset{z\to 1}{Lt}(z-1)\,f(z)=\underset{z\to 1}{Lt}\frac{1-2z}{z(z-2)}=1$$

$$\text{Res. } f(z=2)=\underset{z\to 2}{Lt}(z-2)\,f(z)=\underset{z\to 2}{Lt}\frac{1-2z}{z(z-1)}=-\frac{3}{2}$$

(b) $f(z)=\dfrac{z^2}{z^2+a^2} \Rightarrow \text{Poles}: z=ia, z=-ia$

$$\text{Res.} f(z=ia)=\left(\frac{z^2}{2z}\right)_{z=ia}=\frac{1}{2}ia;\ \text{Res.} f(z=-ia)=\left(\frac{z^2}{2z}\right)_{z=-ia}=-\frac{1}{2}ia$$

(c) $f(z)=z^2e^{1/z}=z^2\left[1+\dfrac{1}{z}+\dfrac{1}{z^2.2!}+\dfrac{1}{z^3.3!}+.......\right] \Rightarrow \text{Poles}: z=0$

$$\text{Res. } f(z=0)=\text{Coefficient of } \frac{1}{z}=\frac{1}{3!}=\frac{1}{6}$$

Example 31: The value of the residue of $\dfrac{\sin z}{z^6}$ is

(a) $-\dfrac{1}{5!}$ (b) $\dfrac{1}{5!}$ (c) $\dfrac{2\pi i}{5!}$ (d) $-\dfrac{2\pi i}{5!}$

Soln: $f(z)=\dfrac{\sin z}{z^6}=\dfrac{1}{z^6}\left[z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\dfrac{z^7}{7!}+........\right]$

$$=\frac{1}{z^5}-\frac{1}{3!}\frac{1}{z^3}+\frac{1}{5!}\frac{1}{z}-......$$

Residue of $f(z)$ = coefficient of $\left(\dfrac{1}{z}\right)=\dfrac{1}{5!}$

Correct option is (b)

5.7 Cauchy Residue Theorem:

If f (z) in single-valued and analytic inside a simple closed curve ‘C, except at a finite number of singular points within C, then

$$\oint_C f(z)dz=2\pi i(\text{sum of the residues at poles within 'C'})$$

Example 32: Evaluate the integral: $\oint_C \frac{4-3z}{z(z-1)(z-3)}dz$ where $|z| = \frac{3}{2}$

Soln: $f(z) = \frac{4-3z}{z(z-1)(z-3)} \Rightarrow \text{Poles}: z=0, z=1, z=3$

But, the given contour is circle centered at the origin and radius 3/2 units.

Therefore, only z = 0 and z = 1 within the contour.

$$I = 2\pi i\left[\text{Res}.f(z=0) + \text{Res}.f(z=1)\right] = 2\pi i\left[\frac{4}{3} - \frac{1}{2}\right] = \frac{5\pi i}{3}$$

Example 33: Evaluate the integral: $\oint_C \frac{e^{2z}+z^2}{(z-1)^5}dz \; where \; |z| = 2$

Soln: $f(z) = \frac{e^{2z}+z^2}{(z-1)^5} \Rightarrow \text{Poles}: z = 1\,(order\ 5)$

$$I = 2\pi i \times \text{Res}.f(z=1) = 2\pi i \times \frac{1}{4!}\frac{d^4}{dz^4}\left[e^{2z}+z^2\right]_{z=1} = 2\pi i \times \frac{2e^2}{3} = \frac{4\pi i e^2}{3}$$

Example 34: The value of the integral $\oint_C \frac{z^3}{z^2-5z+6}dz$, where C is closed contour defined by the equation $2|z|-5=0$, traversed in the anti-clockwise direction, is

(a) $-16\pi i$ (b) $16\pi i$ (c) $8\pi i$ (d) $2\pi i$

Soln: $f(z) = \frac{z^3}{z^2-5z+6}$; Condition of singularity: $z^2-5z+6=0 \;\Rightarrow\; z = 3, 2$ (1st order pole)

Given closed contour: $c: 2|z|-5=0 \;\Rightarrow |z| = \frac{5}{2}$

Only z = 2 i.e. point (2, 0) is within 'c'

$$\text{Res}.f(z)\big|_{z=2} = (z-2)\frac{z^3}{(z-3)(z-2)}\bigg|_{z=2} = -8$$

Given integral: $I = 2\pi i \times (\text{sum of the residues}) = -16\pi i$

Correct answer is (a)

Example 35: The value of the integral $\int_C z^{10}dz$, C is the unit circle with the origin as the centre is

(a) 0 (b) $z^{11}/11$ (c) $2\pi i z^{11}/11$ (d) 1/1

Soln: $f(z) = z^{10}$ is analytic in the entire complex argand plane.

So, according to Cauchy integral theorem,

$$\int_C z^{10}dz = 0$$

Correct option is (a)