$\therefore \quad T=\int_0^x \frac{d}{dt}(mv)\,dx=\int_0^{mv} vd(mv)=\int_0^v vd\left(\frac{m_0 v}{\sqrt{1-v^2/c^2}}\right)$

Integrating by parts, we get

$$T=\frac{m_0 v^2}{\sqrt{1-v^2/c^2}}-m_0\int_0^v \frac{vdv}{\sqrt{1-v^2/c^2}}$$

$$=\frac{m_0 v^2}{\sqrt{1-v^2/c^2}}-m_0\left[-c^2\sqrt{1-v^2/c^2}\right]_0^v$$

$$=\frac{m_0 v^2+m_0 c^2\left(1-\frac{v^2}{c^2}\right)}{\sqrt{1-\frac{v^2}{c^2}}}-m_0 c^2$$

$$=\frac{m_0 c^2}{\sqrt{1-v^2/c^2}}-m_0 c^2=mc^2-m_0 c^2$$

$\therefore \quad \boxed{T=mc^2-m_0 c^2}$

Total energy = kinetic energy + rest mass energy

$\therefore \quad E=T+m_0 c^2=(m-m_0)c^2+m_0 c^2=mc^2$

$$\boxed{E=mc^2}$$

**Total relativistic energy in terms of momentum:**

Sometimes, it is convenient and useful to express the total relativistic energy *E* in terms of the momentum *p*. We have,

$$m^2c^4-m_0^2c^4=m_0^2c^4\left(\frac{1}{1-v^2/c^2}-1\right)$$

$$=m_0^2c^4\frac{v^2/c^2}{1-v^2/c^2}=c^2\frac{m_0^2v^2}{1-v^2/c^2}=c^2p^2 \quad [\because\ p=mv=\gamma m_0 v]$$

$\therefore \quad m^2c^4=p^2c^2+m_0^2c^4 \Rightarrow E^2=p^2c^2+m_0^2c^4$

$\therefore \quad \boxed{E=\sqrt{p^2c^2+m_0^2c^4}}$

**An important theorem:**

Consider a frame $S'$ moving with velocity *v* with respect to frame *S* along its positive *x*-axis. If $u$ and $u'$ be the velocities of a particle in $S$ and $S'$ frames respectively, then

$$\sqrt{1-\frac{u'^2}{c^2}}=\frac{\sqrt{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}}{1-u_xv/c^2}$$

**Proof:** From the relativistic velocity addition theorem,

$$u'_x=\frac{u_x-v}{1-\frac{u_xv}{c^2}};\quad u'_y=\frac{u_y\sqrt{1-v^2/c^2}}{1-\frac{u_xv}{c^2}};\quad u'_z=\frac{u_z\sqrt{1-v^2/c^2}}{1-\frac{u_xv}{c^2}} \qquad \text{... (1)}$$

where, $u_x$'s and $u'_x$'s are the component velocities.

$$\therefore \quad u'^2=u'^2_x+u'^2_y+u'^2_z=\frac{\left(u_x-v\right)^2+\left(u_y^2+u_z^2\right)\left(1-v^2/c^2\right)}{\left(1-u_xv/c^2\right)^2}\text{, using (1)}$$

$$\therefore \quad \frac{u'^2}{c^2}=\frac{\left(\frac{u_x}{c}-\frac{v}{c}\right)^2-\left(\frac{u^2}{c^2}-\frac{u_x^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-u_xv/c^2\right)^2} \qquad \left[\because\ u_y^2+u_z^2=u^2-u_x^2\right]$$

$$\Rightarrow \quad 1-\frac{u'^2}{c^2}=\frac{\left(1-\frac{u_xv}{c^2}\right)^2-\left(\frac{u_x}{c}-\frac{v}{c}\right)^2-\left(\frac{u^2}{c^2}-\frac{u_x^2}{c^2}\right)\left(1-\frac{v^2}{c^2}\right)}{\left(1-u_xv/c^2\right)^2}$$

$$=\frac{\left(1-\frac{v^2}{c^2}\right)-\frac{u^2}{c^2}\left(1-\frac{v^2}{c^2}\right)}{\left(1-u_xv/c^2\right)^2}=\frac{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}{\left(1-u_xv/c^2\right)^2} \qquad \text{... (2)}$$

$$\boxed{1-\frac{u'^2}{c^2}=\frac{\left(1-v^2/c^2\right)\left(1-u^2/c^2\right)}{\left(1-u_xv/c^2\right)^2}}$$

**Transformation formulae for momentum, energy and force:**

Momentum in $S$ frame: The components of momentum are

$$p_x=mu_x=\frac{m_0u_x}{\sqrt{1-u^2/c^2}};\quad p_y=\frac{m_0u_y}{\sqrt{1-u^2/c^2}};\quad p_z=\frac{m_0u_z}{\sqrt{1-u^2/c^2}}$$

where, $u_x$'s are the component velocities.

If $p'_x$, $p'_y$ and $p'_z$ be the momentum components in $S'$ frame