PAPER : IIT-JAM

MATHEMATICS MA–2021

SECTION-A

[Multiple Choice Questions (MCQ)]

Q.1 – Q.10 carry ONE mark each.

1. Let p and t the positive real numbers. Let D_t be the closed disc of radius t centered at (0, 0), i,.e. $D_t = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le t^2\}$. Define

$I(p,t)t = \iint_{D_t} \frac{dxdy}{(p^2 + x^2 + y^2)^p}$ Then $\lim_{t\to\infty} I(p,t)$ is finite

(a) for no value of p (b) onlyif $p < 1$ (c) only if p = 1 (d) only if p >1

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for all $x \in \mathbb{R}$.

$$\int_0^1 f(xt)\,dx = 0 \qquad (*)$$

then

(a) There is an f satisfying (*) that takes both positive and negative values.

(b) There is an f satisfying (*) that is 0 at infinitely many points, but is not identically zero.

(c) f must be identically 0 on the whole of $\mathbb{R}$

(d) There is an f satisfying (*) that is identically 0 on (0, 1) but not identically 0 on the whole of $\mathbb{R}$

3. For every $n \in \mathbb{N}$ let $f_n : \mathbb{R} \to \mathbb{R}$ be a function. From the given choices . pick the statement that is the negation of

"For every $x \in \mathbb{R}$ and for every real number $\in > 0$, there exists an integer $N > 0$ such that $\sum_{i=1}^{p} |f_{N+i}(x)| < \in$ for every integer p > 0".

(a) For every $x \in \mathbb{R}$ and for every real number $\in > 0$, there exists an integer $N > 0$ such that $\sum_{i=1}^{p} |f_{N+i}(x)| < \in$ for every integer p > 0.

(b) For every $x \in \mathbb{R}$ and for every real number $\in > 0$, there does not exist any integer $N > 0$ such that $\sum_{i=1}^{p} |f_{N+i}(x)| \ge \in$ for every integer p > 0.

(c) There exists $x \in \mathbb{R}$ and there exists a real number $\in > 0$ such that for every integer N> 0 and for every integer p > 0 the inequality $\sum_{i=1}^{p} |f_{N+i}(x)| \ge \in$ holds

(d) There exists $x \in \mathbb{R}$ and there exists a real number $\in > 0$ such that for every integer N> 0 there exists an integer $p > 0$ the inequality $\sum_{i=1}^{p} |f_{N+i}(x)| \ge \in$ holds.

4. How many elements of the group $\mathbb{Z}_{50}$ have order 10 ?

(a) 8 (b) 10 (c) 5 (d) 4

5. Let $0<\alpha<1$ be a real numbner of differentiable functions $y:[0,1]\to[0,\infty)$, having continuous derivative on [0, 1] and satisfying

$$\begin{array}{l} y'(t)=(y(t))^{\alpha}, t\in[0,1] \\ y(0)=0 \end{array}$$ is

(a) inifinite (b) exactly one (b) exactly two (d) finite but more than two.

6. Let $n>1$ be an integer, Consider the following two statements for an arbitrary $n\times n$ matrix A with complex entries.

I. If $A^k = I_n$ for some integer $k \geq 1$, then all the eigenvalues of A are k^{th} roots of unity.

II. If for some integer $k \geq 1$, all the eigenvalue of A are k^{th} roots of unity. then $A^k = I_n$.

Then

(a) I is True but II is False (b) neither I nor II is True
(c) both I and II are True (d) I si False but II is True

7. Which one of the following subsets of $\mathbb{R}$ has a non- empty interior ?

(a) The set of all irrational number in $\mathbb{R}$.

(b) The set $\{b\in\mathbb{R} : x^2+bx+1=0 \text{ has distinct roots}\}$

(c) The set $\{a\in\mathbb{R} : \sin(a)=1\}$

(d) The set of all rational numbers in $\mathbb{R}$.

8. Let $P:\mathbb{R}\to\mathbb{R}$ be a continuous function that $P(x)>0$ for all $x\in\mathbb{R}$. Let y be a twice differentiable function on $\mathbb{R}$ satisfying $y''(x)+P(x)y'(x)-y(x)=0$ for all $x\in\mathbb{R}$. Suppose that there exist two real numbers $a,b\,(a<b)$ such that $y(a)=y(b)=0$. Then

(a) $y(x)$ changes sign on (a, b) (b) $y(x)=0\ \forall x\in[a,b]$
(c) $y(x)<0\ \forall x\in(a,b)$ (d) $y(x)>0\ \forall x\in(a,b)$

9. For an integer $k\geq 0$, let P_k denote the vector space of all real polynomials in one variable of degree less than or equal to k. Define a linear transformation $T:P_2\to P_3$ by $Tf(x)=f''(x)+xf(x)$

Which one of the following polynomials is not in the range of T?

(a) $x+x^2$ (b) x^2+x^3+2 (c) $x+1$ (d) $x+x^3+2$

10. Let $f:\mathbb{R}\to\mathbb{R}$ be continuous function satisfying $f(x)=f(x+1)\ \forall x\in\mathbb{R}$. Then

(a) there exists infinitely many $x_0\in\mathbb{R}$ such that $f(x_0+\pi)=f(x_0)$

(b) theere is no $x_0\in\mathbb{R}$ such that $f(x_0+\pi)=f(x_0)$

(c) f is not necessarily bounded above.

(d) there exists a unique $x_0\in\mathbb{R}$ such that $f(x_0+\pi)=f(x_0)$

Q.11 – Q.30 carry TWO marks each.

11. Consider the following statements

I. The group $(\mathbb{Q},+)$ has no proper subgroup of finite index

II. The group $(\mathbb{C}\setminus\{0\},.)$ has no proper subgroup of finite index

Which one of the following statements is true ?

(a) Neither I nor II is True (b) Both I and II are True
(c) II is True but I is False (d) I is True but II is False

12. Let $D \subseteq \mathbb{R}^2$ be defined by $D = \mathbb{R}^2 \setminus \{(x,0) : x \in \mathbb{R}\}$. Consider the function $f : D \to \mathbb{R}$ defined by

$$f(x, y) = x \sin\frac{1}{y}$$

Then

(a) f is a continuous function on D and cannot be extended continuously to any point outside D.

(b) f is discontinuous function on D.

(c) f is a continuous function on D and can be extended continuously to the whole of $\mathbb{R}^2$.

(d) f is a continuous function on D and can be extended continuously to $D \cup \{(0,0)\}$.

13. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cup \{0\} \\ 1 - \frac{1}{p} & \text{if } x - \frac{n}{p}, n \in \mathbb{Z} \setminus \{0\}, p \in \mathbb{N} \text{ and } \gcd(n, p) = 1 \end{cases}$$

then

(a) f is continuous at all $x \in \mathbb{R} \setminus \mathbb{Q}$

(b) f is not continuous at $x = 0$

(c) all $x \in \mathbb{Q} \setminus \{0\}$ are strict local minima for f.

(d) f is continuous at all $x \in \mathbb{Q}$

14. Let y be the solution of

$$(1+x)y''(x) + y'(x) - \frac{1}{1+x}y(x) = 0, \in (-1, \infty)$$

$$y(0) = 1, y'(0) = 0$$

then

(a) y is bounded on $(-1, 0]$

(b) $y(x) \geq 2$ on $(-1, \infty)$

(c) y attained its minimum at $x = 0$

(d) y is bounded on $(0, \infty)$

15. Which one of the following statements is True ?

(a) Exactly half of the elements in any even order subgroup of S_5 must be even permutations

(b) There exists and normal subgroup of S_5 of index 7

(c) There exists a cyclic subgroup of S_5 of order 6.

(d) Any abelian subgroup of S_5 is trivial

16. Which of the following statement is True ?

(a) $(\mathbb{Q}/\mathbb{Z}, +)$ is isomorphic to $(\mathbb{Q}, +)$

(b) $(\mathbb{Q}/\mathbb{Z}, +)$ is isomorphic to $(\mathbb{Q}/2\mathbb{Z}, +)$

(c) $(\mathbb{Z}, +)$ is isomorphic to $(\mathbb{Q}, +)$

(d) $(\mathbb{Z}, +)$ is isomorphic to $(\mathbb{R}, +)$

17. Let $n \geq 2$ be an integer. Let $A : \mathbb{C}^n \to \mathbb{C}^n$ be the linear transformation defined by

$A(z_1, z_2 .... z_n) = (z_n, z_1, z_2 ...., .z_{n-1})$ which one of the follwing statements is true for every $n \geq 2$?

(a) A is nilpotent

(b) All eigen value of A are of modulus 1

(c) A is singular

(d) Every eigenvalue of A is either 0 or 1.

18. Let $f:\mathbb{R}\to\mathbb{R}$ be an infinitely differentiable function such that for all $a,b\in\mathbb{R}$ with $a<b$.

$$\frac{f(b)-f(a)}{b-a}=f'\left(\frac{a+b}{2}\right)$$

Then

(a) f is not a polynomial
(b) f must be a linear polynomial
(c) f must be polynomial of degree less than or equal to 2.
(d) f must be a polynomial of degree greater than 2.

19. Let $M_n(\mathbb{R})$ be the real vector space of all n×n matrices with real entries $n\geq 2$

Let $A\in M_n(\mathbb{R})$. Consider the suspace W of $M_n(\mathbb{R})$ spanned by $\{I_n, A, A^2,....\}$. Then the dimension of W over $\mathbb{R}$ is necessarily

(a) ∞ (b) at most n. (c) n^2 (d) n

20. Consider the family of curves $x^2-y^2=ky$ with parameter $k\in\mathbb{R}$. The equation of the orthogonal trajectory to this family passing through (1, 1) is given by

(a) $x^2+2xy=3$ (b) $x^3+3xy^2=4$ (c) $x^3+2xy^2=3$ (d) $y^2+2x^2y=3$

21. Define $S=\lim\limits_{n\to\infty}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{n^2}\right)$ Then

(a) $S=\frac{3}{4}$ (b) $S=1$ (c) $S=\frac{1}{2}$ (d) $S=\frac{1}{4}$

22. Consider the surface $S=\{(x,y,xy)\in\mathbb{R}^3 : x^2+y^2\leq 1\}$. Let $\vec{F}=y\hat{i}+x\hat{j}+\hat{k}$ if $\hat{n}$ is the continuous unit normal field to the surface S with postive z- component, then $\iint\limits_S \vec{F}\cdot\hat{n}dS$ equals

(a) 2π (b) $\frac{\pi}{2}$ (c) $\frac{\pi}{4}$ (d) π

23. Let $f:[0,1]\to[0,1]$ be a non- contant continuous function such that $f\circ f=f$. Define $E_f=\{x\in[0,1]:f(x)=x\}$. Then

(a) E_f is an interval (b) E_f is empty
(c) E_f is neither open nor closed (d) E_f need not be an interval

24. Let y be a twice differentiable function on $\mathbb{R}$ satisfying

$$y''(x)=2+e^{-|x|}, x\in\mathbb{R}$$
$$y(0)=-1,\quad y'(0)=0$$

Then

(a) There exists an $x_0\in\mathbb{R}$ such that $y(x_0)\geq y(x)$ for all $x\in\mathbb{R}$
(b) $y=0$ has exactly two roots
(c) $y=0$ has exactly one root
(d) $y=0$ has more than two roots

25. Let A be an n×n invertible matrix and C be an n×n nilpotent matrix. If $X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix}$ is a 2n×2n matrix (each n×n) that commutes with the 2n×2n matrix $B = \begin{pmatrix} A & 0 \\ 0 & C \end{pmatrix}$. then

(a) X_{12} and X_{22} are necessarily zero matrices
(b) X_{11} and X_{22} are necessarily zero matrices
(c) X_{12} and X_{21} are necessarily zero matrices
(d) X_{11} and X_{21} are necessarily zero matrices

26. Let g be an element of S_7 such that g commutes with the element (2, 6, 4, 3). The number of such g is

(a) 48
(b) 6
(c) 4
(d) 24

27. Let $f : \mathbb{N} \to \mathbb{N}$ be a bijective map such that $\sum_{n=1}^{\infty} \frac{f(n)}{n^2} < +\infty$. The number of such bijecitve maps is

(a) Zero
(b) infinite
(c) exactly one
(d) finite but more than one

28. Consider the two series

I. $\sum_{n=1}^{\infty} \frac{1}{n^{1+(1/n)}}$ and II. $\sum_{n=1}^{\infty} \frac{1}{n^{2-n^{1/n}}}$

Which one of the following holds ?

(a) Both I and II converge
(b) I diverges and II converges
(c) I converges and II diverges
(d) Both I and II diverge.

29. Let $f : [0,1] \to [0,\infty)$ be a continuous function such that $(f(t))^2 < 1 + 2\int_0^1 f(s)\,ds$, for all $t \in [0,1]$ Then

(a) $f(t) = 1 + t$ for all $t \in [0,1]$
(b) $f(t) < 1 + \frac{t}{2}$ for all $t \in [0,1]$
(c) $f(t) > 1 + t$ for all $t \in [0,1]$
(d) $f(t) < 1 + t$ for all $t \in [0,1]$

30. Let G be a finte abelian group of odd order. Consider the following two statements:

I. The map $f : G \to G$ define by $f(g) = g^2$ is a group isomorphism

II. The product $\prod_{g \in G} g = e$

(a) Both I and II are True
(b) Neither I nor II is True
(c) II is True but I is False
(d) I is Ture but II is False

SECTION-B

[Multiple Select Questions (MSQ)]

Q.01 – Q.10 carry TWO marks each.

1. Let G be a finite group of order 28. Assume that G contain a subgroup of order 7. Which of the following statements is/are True ?

(a) G contains atleast two subgroups of order 7
(b) G contains normal subgroups of order 7
(c) G contains a unique subgroups of order 7
(d) G contains no normal subgroups of order 7

2. Let $f : (a,b) \to \mathbb{R}$ be a differentiable function on (a, b). Which of the following statements is/are True?

(a) If $f'(x_0) > 0$ for some $x_0 \in (a,b)$, then there exists a $\delta > 0$ such that $f(x) > f(x_0)$ for all $x \in (x_0, x_0 + \delta)$

(b) If $f'(x_0) > 0$ for some $x_0 \in (a,b)$, then f is increasing in a neighbourhod of x_0.

(c) $f' > 0$ in (a, b) implies that f is increasing in (a, b)

(d) f is increasing in (a,b) implies that $f' > 0$ in (a, b)

3. Let V be a finite dimensional vector space and $T:V \to V$ be a linear transformation. Let $R(T)$ denote the range of T and N(T) denote the null space $\{v \in V : Tv = 0\}$ of T. If rank (T) = rank (T^2), then which of the following is/are necessarily true ?

(a) $N(T) = \{0\}$ (b) $N(T) = N(T^2)$ (c) $N(T) \cap R(T) = \{0\}$ (d) $R(T) = R(T^2)$

4. Consider the four function from $\mathbb{R}$ to $\mathbb{R}$: $f_1(x) = x^4 + 3x^3 + 7x + 1$, $f_2(x) = x^3 + 3x^3 + 4x$, $f_3(x)$ = arctan (x) and $f_4(x) = \begin{cases} x & \text{if } x \notin \mathbb{Z} \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$

Which of the following subsets of $\mathbb{R}$are open ?

(a) The range of f_4 (b) The range of f_2 (c) The range of f_1 (d) The range of f_3

5. Which of the following subsets of R is/are connected ?

(a) The set $\{x \in \mathbb{R} : x^3 + x + 1 \geq 0\}$ (b) The set $\{x \in \mathbb{R} : x \text{ is irrational}\}$

(c) The set $\{x \in \mathbb{R} : x^3 - 2x + 1 \geq 0\}$ (d) The set $\{x \in \mathbb{R} : x^3 - 1 \geq 0\}$

6. Consider the two function $f(x, y) = x + y$ and $g(x, y) = xy - 16$ defined on $\mathbb{R}^2$. Then

(a) The function g has a global extreme value at (0, 0 subject to the condition $f = 0$

(b) The function g has a global extreme value subject to the condition $f = 0$

(c) The function f has no global extreme value subject to the condition $g = 0$

(d) The function f attains global extreme value at (4, 4) and (–4, –4) subject to the condition $g = 0$

7. Let $D = \mathbb{R}^2 \setminus \{(0,0)\}$, Consider the two functions $u, v : D \to \mathbb{R}$ defined by

$u(x, y) = x^2 - y^2$ and $u(x, y) = xy$

Consider the gradients ∇u and ∇v of the functions u and v, respectively. Then

(a) ∇u and ∇v are perpendicular at each point (x, y) of D

(b) ∇u and ∇v are parallel at each point (x, y) of D

(c) ∇u and ∇v are each point (x, y) of D span $\mathbb{R}^2$

(d) ∇u and ∇v do not exist at somepoint (x, y) of D

8. Let $m > 1$ and $n > 1$ be integers. Let A be an $m \times n$ matrix such that for some $m \times 1$ matrix b_1, the equation $Ax = b_1$ has infinitely many solutions. Let b_2 denote an $m \times 1$ matrix different from b_1, then $Ax = b_2$ has

(a) Finitely many solutions for some b_2. (b) No solution for some b_2.

(c) infinitely many solutions for some b_2. (d) A unique solution for some b_2.

9. Consider the equation $x^{2021} + x^{2020} + ... + x - 1 = 0$ Then

(a) exactly one real root is positive (b) no real roots is positive

(c) all real roots are positive (d) exactly one real root is negative

10. Let $f : \mathbb{R} \to \mathbb{R}$ be a funtion with the property that for every $y \in \mathbb{R}$. The value of the expression $\sup_{x \in \mathbb{R}} [xy - f(x)]$ is finite. Define $g(y) = \sup_{x \in \mathbb{R}} [xy - f(x)]$ for $y \in \mathbb{R}$. Then

(a) f must satisfy $\lim_{|x| \to \infty} \frac{f(x)}{|x|} = +\infty$ (b) g is odd if f is even

(c) g is even if f is even (d) f must satisfy $\lim_{|x| \to \infty} \frac{f(x)}{|x|} = -\infty$

CAREER ENDEAVOUR

SECTION-C

[Numerical Answer Type (NAT)]

Q.01 – Q.10 carry ONE mark each.

1. The number of group homomorphisms from the group $\mathbb{Z}_4$ to the group S_3 is ______

2. Consider the subset $S = \{(x,y): x^2+y^2>0\}$ of $\mathbb{R}^2$. Let

$$P(x,y) = \frac{y}{x^2+y^2} \text{ and } Q(x,y) - \frac{x}{x^2+y^2}$$

For $(x,y)\in S$. If C denotes the unit circle traversed in the counter-clockwise direction, then the value of

$\frac{1}{\pi}\int_C (Pdx+Qdy)$ is _______

3. Let $y:\left(\frac{9}{10},3\right)\to\mathbb{R}$ be a differentiable function satisfying

$(x-2y)\frac{dy}{dx}+(2x+y)=0, x\in\left(\frac{9}{10},3\right)$, and $y(1)=1$. then $y(2)$ equals ________

4. Consider the set $A=\{a\in\mathbb{R}: x^2=a(a+1)(a+2) \text{ has a real root}\}$. The number of connected components of A is ____________.

5. The value of $\lim\limits_{n\to\infty}\left(3^n+5^n+7^n\right)^{\frac{1}{n}}$ is _______.

6. Let $B=\{(x,y,z)\in\mathbb{R}^3: x^2+y^2+z^2\le 1\}$ and define $u(x,y,z)=\sin\left(\left(1-x^2-y^2-z^2\right)^2\right)$ for $(x,y,z)\in B$. Then the value of $\iiint_B\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial^2 y}+\frac{\partial^2 u}{\partial z^2}\right)dxdydz$ is ______

7. The number of cycles of length 4 in S_6 is______

8. The value of $\frac{\pi}{2}\lim\limits_{n\to\infty}\cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{8}\right)....\cos\left(\frac{\pi}{2^{n+1}}\right)$ is _______

9. Let V be the real vector space of all continuous function $f:[0,2]\to\mathbb{R}$ such that the restriction of *f* to the interval [0, 1] is a polynomial of degree less than or equal to 2, the restriction of f to the interval [1, 2] is a polynomial of degree less than or equal to 3 and f (0)= 0. Then the dimension of V is equal to ______

10. Let $\vec{F}=(y+1)e^y\cos(x)\hat{i}+(y+2)e^y\sin(x)\hat{j}$ be a vector field in $\mathbb{R}^2$ and C be continuously diffferentiable path with the starting point (0, 1) and the end point $\left(\frac{\pi}{2},0\right)$. Then $\int_C \vec{F}\cdot d\vec{r}$ equals_______

Q.11 – Q.20 carry TWO marks each.

11. Consider those continuous functions $f:\mathbb{R}\to\mathbb{R}$ that have the property that given any $x\in\mathbb{R}$. $f(x)\in\mathbb{Q}$ if

$f(x+1)\in\mathbb{R}\setminus\mathbb{Q}$. The number of such functions is ____________

12. The number of elements of order two in the group S_4 is equal to ________

13. Let $A=\begin{pmatrix}2 & -1 & 3\\ 2 & -1 & 3\\ 3 & 2 & -1\end{pmatrix}$. Then the largest eigenvalue of A is______________

14. The least possible value of k, accurate up to two decimal place, for which the following problem

$y''(t)+2y'(t)+ky(t)=0, t\in\mathbb{R}$

$y(0)=0, y(1)=0, y(1/2)=1$ has a solution is ______________

15. Let $A=\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$. Consider the linear map T_A from the real vector space $M_4(\mathbb{R})$ to itself defined by $T_A(X)=AX-XA$, for all $X\in M_4(\mathbb{R})$. The dimension of the range of T_A is_______

16. The determinant of the matrix $\begin{pmatrix}2021 & 2020 & 2020 & 2020\\ 2021 & 2021 & 2020 & 2020\\ 2021 & 2021 & 2021 & 2020\\ 2021 & 2021 & 2021 & 2021\end{pmatrix}$ is _______

17. The largest positive number a such that $\int_0^5 f(x)\,dx+\int_0^3 f^{-1}(x)\,dx\geq a$ for every strictly increasing surjective continuous function $f:[0,\infty)\to[0,\infty)$ is __________

18. Define the sequence $S_n=\begin{cases}\dfrac{1}{2^n}\displaystyle\sum_{j=0}^{n-2}2^{2j} & \text{if } n>0 \text{ is even}\\ \dfrac{1}{2^n}\displaystyle\sum_{j=0}^{n-1}2^{2j} & \text{if } n>0 \text{ is odd}\end{cases}$

Define $\sigma_m=\dfrac{1}{m}\displaystyle\sum_{n=1}^{m}s_n$. The number of limit points of the sequence $\{\sigma_m\}$ is _______

19. Let S be the surface defined by $\{(x,y,z)\in\mathbb{R}^3 : z=1-x^2-y^2, z\geq 0\}$. Let $\vec{F}=-y\hat{i}+(x-1)\hat{j}+z^2\hat{k}$ and $\hat{n}$ be the continuous unit normal field to the surface S with positive z-component. Then the value of $\dfrac{1}{\pi}\displaystyle\iint_S(\nabla\times\vec{F})\cdot\hat{n}\,dS$ is _________

20. The value of $\displaystyle\lim_{n\to\infty}\int_0^1 e^{x^2}\sin(nx)\,dx$ is________________

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CAREER ENDEAVOUR