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Definite Integral & its Application

1. DEFINITE INTEGRALS

1.1 INTRODUCTION

Consider a function $f(x)$ whose indefinite integral is $F(x)+C$.

i.e., $\int f(x)\,dx = F(x)+C$

Also consider the integral $\int_a^b f(x)\,dx$, which is known as a definite integral and $x=a,\ x=b$ are called the lower and upper limits of integration.

The relationship between the definite integral $\int_a^b f(x)\,dx$ and the indefinite integral $F(x)$ is :

$$\int_a^b f(x)\,dx = F(b)-F(a)$$

This formula comes from The fundamental Theorem of calculus (FTC). This formula can be used only if the function $f(x)$ is continuous at all points in the interval $[a, b]$.

Ex.1: Evaluate :

(i) $\int_1^3 x^2\,dx$ (ii) $\int_0^{\pi/2} \sin x\,dx$

Soln. (i) $\int_1^3 x^2\,dx = \left|\frac{x^3}{3}\right|_1^3 = \frac{1}{3}(3^3-1^3)=\frac{26}{3}$ (ii) $\int_0^{\pi/2} \sin x\,dx = \left|-\cos x\right|_0^{\pi/2} = -(\cos \pi/2 - \cos 0)=1$

Ex.2: Evaluate : $\int_0^{\pi/2} \sin^3 x \cos x\,dx$

Soln. Let $I=\int_0^{\pi/2} \sin^3 x \cos x\,dx$

Let $\sin x = t \ \Rightarrow\ \cos x\,dx = dt$

For $x=\frac{\pi}{2}, t=1$ and for $x=0,\ t=0$

$$\Rightarrow\ I=\int_0^1 t^3\,dt = \left|\frac{t^4}{4}\right|_0^1 = \frac{1}{4}.$$

Note: Whenever we use substitution in a definite integral, we have to change the limits corresponding to the change in the variable of the integration.

In this example we have applied F.T.C to calculate the definite integral. F.T.C is applicable here since $\sin^3 x \cos x$ (integrand) is a continuous function in the interval $[0, \pi/2]$.

PROPERTIES OF DEFINITE INTEGRALS

2.1 BASIC PROPERTIES OF DEFINITE INTEGRALS

Property - 1 : $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a < c < b$

Property - 2 : $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Property - 3 : $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

Ex.3: Evaluate : $\int_{-4}^{3} \left|x^2 - 4\right| dx$

Soln. $\int_{-4}^{3} \left|x^2-4\right| dx = \int_{-4}^{-2} \left|x^2-4\right| dx + \int_{-2}^{+2} \left|x^2-4\right| dx + \int_{2}^{3} \left|x^2-4\right| dx$

$$= \int_{-4}^{-2} (x^2-4)\,dx + \int_{-2}^{2} (4-x^2)\,dx + \int_{2}^{3} (x^2-4)\,dx$$

[as $\left|x^2-4\right| = 4 - x^2$ in $[-2, 2]$ and $x^2 - 4$ in other intervals]

$$= \left|\frac{x^3}{3} - 4x\right|_{-4}^{-2} + \left|4x - \frac{x^3}{3}\right|_{-2}^{2} + \left|\frac{x^3}{3} - 4x\right|_{2}^{3}$$

$$= \left(-\frac{8}{3} + 8\right) - \left(-\frac{64}{3} + 16\right) + \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) + \left(\frac{27}{3} - 12\right) - \left(\frac{8}{3} - 8\right) = \frac{71}{3}.$$

2.2 PROPERTIES OF DEFINITE INTEGRALS :

Property - 4 : $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

Ex.4: Evaluate: $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$

Soln. Let $I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ ... (i)

Using property - 4, we have :

$$I = \int_0^{\pi/2} \frac{\sqrt{\sin(\pi/2 - x)}}{\sqrt{\sin(\pi/2 - x)} + \sqrt{\cos(\pi/2 - x)}}\,dx$$

$$I = \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\cos x} + \sqrt{\sin x}} dx \qquad \text{... (ii)}$$

Adding (i) and (ii), we get

$$2I = \int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx + \int_0^{\pi/2} \frac{\sqrt{\cos x}}{\sqrt{\sin x} + \cos x} dx$$

$$\Rightarrow \quad 2I = \int_0^{\pi/2} \frac{\sqrt{\sin x} + \sqrt{\cos x}}{\sqrt{\sin x} + \sqrt{\cos x}} dx \Rightarrow \quad 2I = \int_0^{\pi/2} dx = \frac{\pi}{2} \quad \Rightarrow \quad I = \frac{\pi}{4}.$$

Ex.5: If $f(a-x) = f(x)$, then show that $\int_0^a x\, f(x)\, dx = \frac{a}{2} \int_0^a f(x)\, dx$.

Soln. Let $I = \int_0^a x\, f(x)\, dx \Rightarrow \quad I = \int_0^a (a-x)\, f(a-x)\, dx$

$$\Rightarrow \quad I = \int_0^a (a-x)\, f(x)\, dx \qquad [\text{using } f(x) = f(a-x)]$$

$$\Rightarrow \quad I = \int_0^a a\, f(x)\, dx - \int_0^a x\, f(x)\, dx \quad \Rightarrow \quad I = a\int_0^a f(x)\, dx - I$$

$$\Rightarrow \quad 2I = a\int_0^a f(x)\, dx \qquad \Rightarrow \quad I = \frac{a}{2}\int_0^a f(x)\, dx$$

Hence proved.

2.3 PROPERTIES OF DEFINITE INTEGRALS :

Property - 5 : $\int_0^{2a} f(x)\, dx = \int_0^a f(x)\, dx + \int_0^a f(2a-x)\, dx$

Property - 6 : $\int_0^{2a} f(x)\, dx = 0$ if $f(2a-x) = -f(x)$; $\int_0^{2a} f(x)\, dx = 2\int_0^a f(x)\, dx$ if $f(2a-x) = f(x)$

Ex.6: Evaluate : $\int_0^{\pi} \frac{x}{1+\cos^2 x} dx$

Soln. Let $I = \int_0^{\pi} \frac{x}{1+\cos^2 x} dx$ ... (i)

$$\Rightarrow \quad I = \int_0^{\pi} \frac{(\pi - x)}{1+\cos^2(\pi - x)} dx \qquad [\text{using property - 4}] \qquad \text{... (ii)}$$

Adding (i) and (ii), we get :

$$\Rightarrow \quad 2I = \int_0^{\pi} \frac{\pi}{1+\cos^2 x} dx$$

$$\Rightarrow \quad I = \frac{\pi}{2}\int_0^{\pi} \frac{dx}{1+\cos^2 x} = \frac{2\pi}{2}\int_0^{\pi/2} \frac{dx}{1+\cos^2 x} \qquad [\text{using property - 6}]$$

Divide N^r and D^r by $\cos^2 x$ to get :

$$\Rightarrow \quad I = \pi \int_0^{\pi/2} \frac{\sec^2 x}{\sec^2 x + 1} dx$$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$ $\quad [\sec^2 x = 1 + \tan^2 x]$

For $x = \pi/2,\ t \to \infty$ and for $x = 0,\ t = 0$

$$\Rightarrow \quad I = \pi \int_0^{\infty} \frac{dt}{2+t^2} \quad \Rightarrow \quad I = \left| \frac{\pi}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} \right|_0^{\infty} = \frac{\pi}{\sqrt{2}} \times \frac{\pi}{2} = \frac{\pi^2}{2\sqrt{2}}.$$

2.4 PROPERTIES OF DEFINITE INTEGRALS : (contd...)

Property - 7 : $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Property - 8 : $\int_{-a}^{a} f(x)\,dx = \int_0^a [f(x) + f(-x)]\,dx$

$$= \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) \text{ is even i.e. } f(-x) = f(x) \\ 0 & \text{if } f(x) \text{ is odd i.e. } f(-x) = -f(x) \end{cases}$$

Property - 9 : $\int_0^{nT} f(x)\,dx = n\int_0^T f(x)\,dx$, where $f(x)$ is a periodic function with period T and n is an integer.

Ex.7: Evaluate : $\int_{-1}^{+1} \log\left(\frac{2-x}{2+x}\right) \sin^2 x\,dx$

Soln. Let $f(x) = \log\left(\frac{2-x}{2+x}\right)\sin^2 x \Rightarrow f(-x) = \log\left(\frac{2+x}{2-x}\right)\sin^2(-x)$

$$\Rightarrow \quad f(-x) = \log\left(\frac{2-x}{2+x}\right)^{-1} \sin^2 x = -\log\left(\frac{2-x}{2+x}\right)\sin^2 x = -f(x) \Rightarrow \quad f(x) \text{ is an odd function.}$$

$$\therefore \quad \int_{-1}^{1} \log\left(\frac{2-x}{2+x}\right)\sin^2 x\,dx = 0.$$

Ex.8: Determine a positive integer $n \le 5$ such that $\int_0^1 e^x (x-1)^n\,dx = 16 - 6e$

Soln. Let $I_n = \int_0^1 e^x (x-1)^n\,dx$ $\quad$ [using integration by parts]

$$I_n = \left[(x-1)^n \int e^x dx\right]_0^1 - \int_0^1 e^x n(x-1)^{n-1} dx$$

$$I_n = 0 - (-1)^n - n\int_0^1 e^x (x-1)^{n-1} dx$$

$$I_n = -(-1)^n - n\,I_{n-1} \qquad \text{... (i)}$$

Also $I_0 = \int_0^1 e^x (x-1)^0\,dx = e-1$

$\Rightarrow \quad I_1 = 1 - I_0 = 1-(e-1) = 2-e$ [using (i)]

$\Rightarrow \quad I_2 = -1 - 2I_1 = -1 - 2(2-e) = -5+2e$

$\Rightarrow \quad I_3 = 1 - 3I_2 = 1 - 3(-5+2e) = 16-6e$

$\Rightarrow$ Hence for $n=3$, $\int_0^1 e^x (x-1)^n\,dx = 16-6e$

2.5 PROPERTIES OF DEFINITE INTEGRALS :

Property - 10 :

$$\int_0^\infty f(x)\,dx = \lim_{n\to\infty}\int_0^n f(x)\,dx \quad ; \quad \int_{-\infty}^a f(x)\,dx = \lim_{n\to-\infty}\int_n^a f(x)\,dx$$

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx = \lim_{a\to-\infty}\int_a^0 f(x)\,dx + \lim_{b\to\infty}\int_0^b f(x)\,dx$$

The integral $\int_{-\infty}^{\infty} f(x)\,dx$ converges if both of the above integrals converge.

Property - 11 : $\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$.

Property - 12 : If the function $f(x)$ and $g(x)$ are defined on $[a, b]$ and differentiable at all points $x \in [f(a), g(a)]$, then $\frac{d}{dx}\left[\int_{f(x)}^{g(x)} h(t)\,dt\right] = h[g(x)]\,g'(x) - h[f(x)]\,f'(x)$ (Lebnitz rule)

Property - 13 : If $f(x) \ge g(x)$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$.

Put $g(x) = 0$ for all $x \in [a, b]$ in above property to get another useful property, i.e.

If $f(x) \ge 0$ for all $x \in [a, b]$, then $\int_a^b f(x)\,dx \ge 0$.

Ex.9: If $f(x) = \int_{x^2}^{x^3} \frac{1}{\log t}\,dt \;\; t > 0$, then find $f'(x)$.

Soln. Using the property - 12,

$$f'(x) = \frac{1}{\log(x^3)}\frac{d}{dx}(x^3) - \frac{1}{\log x^2}\frac{d}{dx}(x^2) \quad\Rightarrow\quad f'(x) = \frac{3x^2}{3\log x} - \frac{2x}{2\log x} = \frac{x^2-x}{\log x}.$$

Ex.10: Find the points of local minimum and local maximum of the function $\int_0^{x^2} \frac{t^2-5t+4}{2+e^t}\,dt$.

Soln. Let $y = \int_0^{x^2} \frac{t^2-5t+4}{2+e^t} = \int_0^{x^2} \frac{(t-1)(t-4)}{2+e^t}\,dt$

For the points of extremum, $\frac{dy}{dx}=0$ [using property - 12]

$$\left[\frac{(x^2-1)(x^2-4)}{2+e^{x^2}}\right](2x)=0$$

$\Rightarrow \quad x=0 \quad \text{or} \quad x^4-5x^2+4=0$

$\Rightarrow \quad x=0 \quad \text{or} \quad (x-1)(x+1)(x-2)(x+2)=0$

$\Rightarrow \quad x=0 \quad , \quad x=\pm 1$ and $x=\pm 2$

With the help of first derivative test, check yourself that $x=-2, 0, 2$ are points of local minimum and $x=-1,$ are points of local maximum.

IMPORTANT RESULT

3.1 DEFINITE INTEGRAL AS A LIMIT OF A SUM

We can express definite integral as a limit of the sum of a certain number of terms. Let $f(x)$ be a continuous function in the interval $[a, b]$. Divide $[a, b]$ interval into n equal parts such that width of each part is h.

$\Rightarrow \quad nh=b-a$

The definite integral of a function $f(x)$ in the interval $[a, b]$ can be defined as :

$$\int_a^b f(x)\,dx=\lim_{n\to\infty} h\left[f(a+h)+f(a+2h)+\cdots+f(a+nh)\right] \qquad \text{where } nh=b-a$$

$$=\lim_{\substack{n\to\infty\\ h\to 0}} h\sum_{r=1}^{n} f(a+rh) \qquad \text{where } nh=b-a$$

With the help of this formula, we can evaluate some simple definite integrals. The process of finding definite integrals with the use of above formula is known as definite integral as a limit of a sum or definite integral by first principle.

3.2 SUMMATION OF SERIES WITH HELP OF DEFINITE INTEGRALS

Consider the "limit of a sum" formula defined in the previous section i.e.,

$$\int_a^b f(x)\,dx=\lim_{n\to\infty} h\sum_{r=1}^{n} f(a+rh) \qquad \text{where } nh=b-a \qquad \text{... (i)}$$

Put $a=0$ and $b=1$, $\Rightarrow \; nh=1 \;\Rightarrow\; n=1/n.$

Put $a=0$ and $b=1$ and $h=1/n$ in (i) to get :

$$\int_0^1 f(x)\,dx=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) \qquad \text{... (ii)}$$

With the help of formula (ii), we can evaluate the sum to infinite terms of certain series.

Working Rule : First of all express the given series in the form

$\frac{1}{n}\sum f\left(\frac{r}{n}\right)$ and then replace the integral sign $\int$ for $\sum$ and $\frac{r}{n}$ by x.

The lower and upper limit of integration are the values of $\lim_{n\to\infty}\left(\frac{r}{n}\right)$ for the least and the greatest values of r respectively.

3.3 ESTIMATION OF A DEFINITE INTEGRAL

If $f(x)$ is a function defined in the integral $[a, b]$ then $m(b-a) \le \int_a^b f(x)\,dx \le M\,(b-a)$, where m is the least and M is the greatest value of the function $f(x)$ in the interval $[a, b]$.

3.4 MEAN VALUE THEOREM OF DEFINITE INTEGRALS

If the function $f(x)$ is continuous in the interval $[a, b]$, then $\int_a^b f(x)\,dx = f(c)\,(b-a)$, where $a < c < b$.

3.5 TWO USEFUL FORMULAE

1. If n be a positive integer, then :

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \begin{cases} \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdots\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2}, & \text{when } n \text{ is even} \\ \dfrac{n-1}{n}\cdot\dfrac{n-3}{n-2}\cdot\dfrac{n-5}{n-4}\cdots\dfrac{4}{5}\cdot\dfrac{2}{3}\cdot 1, & \text{when } n \text{ is odd} \end{cases}$$

2. $$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \int_0^{\pi/2} \sin^n x \cos^m x\,dx$$

$$= \begin{cases} \dfrac{(m-1)\cdot(m-3).....(1 \text{ or } 2)\;(n-1)\cdot(n-3).....(1 \text{ or } 2)}{(m+n)\cdot(m+n-2).....(1 \text{ or } 2)}\,\dfrac{\pi}{2} & \text{when both } m \,\&\, n \text{ even integer} \\ \dfrac{(m-1)\cdot(m-3).....(1 \text{ or } 2)\;(n-1)\cdot(n-3).....(1 \text{ or } 2)}{(m+n)\cdot(m+n-2).....(1 \text{ or } 2)}, & \text{otherwise} \end{cases}$$

(Walli's Formula)

Ex.11: Evaluate : $\int_a^b x^2\,dx$ using limit of a sum formula.

Soln. Let

$$\Rightarrow \quad I = \int_a^b x^2\,dx = \lim_{\substack{n\to\infty \\ h\to 0}} h\left[(a+h)^2 + (a+2h)^2 + \cdots + (a+nh)^2\right]$$

$$\Rightarrow \quad I = \lim_{\substack{n\to\infty \\ h\to 0}} \left[nha^2 + \frac{2ah^2 n(n+1)}{2} + \frac{h^3 n(n+1)\,(2n+1)}{6}\right]$$

Using $nh = b - a$, we get :

$$\Rightarrow \quad I = \lim_{n\to\infty} \left[a^2(b-a) + a(b-a)^2\left(1+\frac{1}{n}\right) + (b-a)^3\,\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$\Rightarrow \quad I = a^2(b-a) + a(b-a)^2 + \frac{(b-a)^3}{6}\,(2)$$

$$\Rightarrow \quad I=(b-a)\left[a^2+ab-a^2+\frac{b^2+a^2-2ab}{3}\right] \Rightarrow \quad I=\frac{(b-a)}{3}\left[a^2+b^2+ab\right]=\frac{b^3-a^3}{3}.$$

Ex.12: Evaluate the following sum. $S=\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}\right]$.

Soln. $\Rightarrow \quad S=\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+n}\right]$

$$\Rightarrow \quad S=\lim_{n\to\infty}\frac{1}{n}\left[\frac{n}{n+1}+\frac{n}{n+2}+\cdots+\frac{n}{2n}\right] \Rightarrow \quad S=\lim_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+1/n}+\frac{1}{1+2/n}+\cdots+\frac{1}{1+n/n}\right]$$

$$\Rightarrow \quad S=\lim_{n\to\infty}\frac{1}{n}\left[\sum_{r=1}^{n}\frac{1}{1+r/n}\right] \Rightarrow \quad S=\int_0^1\frac{1}{1+x}dx \Rightarrow \quad S=\left|\log(1+x)\right|_0^1=\log 2.$$

Ex.13: Find the sum of the series : $\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{6n}\right)$.

Soln. Let $S=\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{6n}\right)$

Take $1/n$ common from the series i.e., $S=\lim_{n\to\infty}\frac{1}{n}\left(\frac{1}{1+1/n}+\frac{1}{1+2/n}+\cdots+\frac{1}{1+5n/n}\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{5n}\frac{1}{1+r/n}$

For the definite integral,

Lower limit $=a=\lim_{n\to\infty}\left(\frac{r}{n}\right)=\lim_{n\to\infty}\frac{1}{n}=0$

Upper limit $=b=\lim_{n\to\infty}\left(\frac{r}{n}\right)=\lim_{n\to\infty}\frac{5n}{n}=5$

Therefore, $S=\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{5n}\frac{1}{1+(r/n)}=\int_0^5\frac{dx}{1+x}=\ell n\left|1+x\right|_0^5=\ell n\, 6-\ell n\, 1=\ell n\, 6.$

Ex.14: Show that : $1\le\int_0^1 e^{x^2}dx\le e$

Soln. Using the result given in section 3.3, $m(1-0)\le\int_0^1 e^{x^2}dx\le M(1-0)$ ... (i)

Let $f(x)=e^{x^2}$

$\Rightarrow \quad f'(x)=2xe^{x^2}=0 \Rightarrow x=0$

Apply first derivative test to check that there exists a local minimum at $x=0$.

$\Rightarrow \quad f(x)$ is an increasing function in the interval [0, 1]

$\Rightarrow \quad m=f(0)=1$ and $M=f(1)=e^1=e$

Substituting the values of m and M in (i), we get $(1-0)\le\int_0^1 e^{x^2}dx\le e(1-0)$

$\Rightarrow \quad 1\le\int_0^1 e^{x^2}dx\le e$ Hence proved.

Ex.15: Find the sum of the following series as $n \to \infty$,

$$\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+.....+\frac{n}{n^2+n^2}$$

Soln. Let $S = \lim_{n\to\infty}\left[\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\frac{n}{n^2+3^2}+.....+\frac{n}{n^2+n^2}\right]$

$$= \lim_{n\to\infty}\left[\sum_{r=1}^{n}\frac{n}{n^2+r^2}\right] = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=1}^{n}\frac{n^2}{n^2+r^2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+(r/n)^2}\right] = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

For the corresponding definite integral,

Lower limit $= \lim_{n\to\infty}\left(\frac{r}{n}\right) = \lim_{n\to\infty}\frac{1}{n} = 0 \quad (r = 1 \text{ for first term})$

Upper limit $= \lim_{n\to\infty}\left(\frac{n}{n}\right) = 1 \quad (r = n \text{ for the last term})$

$$\therefore \quad S = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=1}^{n}\frac{1}{1+(r/n)^2}\right] = \int_0^1 \frac{1}{1+x^2}dx = \left[\tan^{-1}x\right]_0^1 = \frac{\pi}{4}$$

Ex.16: Evaluate $\lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+.....+\frac{1}{64n}\right]$

Soln. Let $S = \lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+.....+\frac{1}{64n}\right] = \lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+.....+\frac{n^2}{(n+3n)^3}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{3n}\frac{n^2}{(n+r)^3} = \lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=1}^{3n}\frac{1}{(1+r/n)^3}\right]$$

For the corresponding definite integral,

Upper limit $= \lim_{n\to\infty}\frac{3n}{n} = 3$

Lower limit $= \lim_{n\to\infty}\frac{1}{n} = 0$

$$\therefore \quad \lim_{n\to\infty}\left[\frac{1}{n}\sum_{r=1}^{3n}\frac{1}{(1+r/n)^3}\right] = \int_0^3 \frac{1}{(1+x)^3}dx = -\frac{1}{2}\left.\frac{1}{(1+x)^2}\right|_0^3 = -\frac{1}{2}\left[\frac{1}{16}-1\right] = \frac{15}{32}.$$

Ex.17: Evaluate : $\lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\frac{1}{\sqrt{4n^2-9}}+\cdots+\frac{1}{\sqrt{3n^2}}\right]$

Soln. $L = \lim_{n\to\infty}\left[\frac{1}{\sqrt{4n^2-1}}+\frac{1}{\sqrt{4n^2-4}}+\frac{1}{\sqrt{4n^2-9}}+\cdots+\frac{1}{\sqrt{3n^2}}\right]$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{\sqrt{4n^2-r^2}} = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{(1-0)}{n}\frac{1}{\sqrt{4-\left(0+r\left(\frac{1-0}{n}\right)\right)^2}}$$

Which is of the form $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{b-a}{n}f\left(a+r\left(\frac{b-a}{n}\right)\right)$

Here $b=1,\ a=0$ and $f(x)=\frac{1}{\sqrt{4-x^2}}$

So, $L=\int_0^1\frac{dx}{\sqrt{4-x^2}}=\sin^{-1}\frac{x}{2}\Big|_0^1=\frac{\pi}{6}.$

Ex.18: Evaluate : $\lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots+\frac{1}{64n}\right]$

Soln. $L=\lim_{n\to\infty}\left[\frac{n^2}{(n+1)^3}+\frac{n^2}{(n+2)^3}+\cdots+\frac{1}{64n}\right]=\lim_{n\to\infty}\sum_{r=1}^{3n}\frac{n^2}{(n+r)^3}$

Put $3n=m$, we get, $L=\lim_{n\to\infty}\sum_{r=1}^{m}\frac{m^2/9}{\left(\frac{m}{3}+r\right)}=\lim_{n\to\infty}\sum_{r=1}^{m}\frac{3}{m}\left(\frac{1}{1+\frac{3r}{m}}\right)^3=\int_0^3\frac{dx}{(1+x)^3}=\frac{-1}{2(1+x)^2}\Bigg|_0^3=\frac{15}{32}.$

5. DEFINITE INTEGRALS DEPENDENT ON PARAMETERS

Let given integral is $I(\alpha)=\int_a^b f(x,\alpha)\,dx,\ a\le x\le b$

Here,
(i) x is variable of integration
(ii) α is a parameter independent of x
(iii) a and b are constants
If this integral can not be solved directly then partially differentiating both sides w.r.t. α

$$\frac{dI}{d\alpha}=\int_a^b\frac{\partial}{\partial\alpha}\left(f(x,\alpha)\right)dx$$

Now, $\frac{\partial}{\partial\alpha}f(x,\alpha)$ is easily integrable $\frac{dI}{d\alpha}=g(\alpha)$

$I=\int g(\alpha)\,d\alpha\ ;\ I=h(\alpha)+c$

c can be found by initial integral.

Ex.19: Evaluate : $\int_0^1\frac{x^b-1}{\log_e x}$, '$b$' being parameter

Soln. Let $I(b)=\int_0^1\frac{x^b-1}{\log_e x}dx=\frac{dI(b)}{db}=\int_0^1\frac{x^b\log_e x}{\log_e x}dx+0-0$ (using modified Leibnitz theorem)

$$=\int_0^1 x^b\,dx=\frac{x^{b+1}}{b+1}\Bigg]_0^1=\frac{1}{b+1}$$

$I(b) = \log_e(b+1) + c$

$b = 0 \;\Rightarrow\; I(0) = 0$

$c = 0 \;\;\therefore I(b) = \log_e(b+1)$

Ex.20: Evaluate : $\int_0^1 \frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}}\,dx$, 'a' being parameter

Soln. Let $I(a) = \int_0^1 \frac{\tan^{-1}(ax)}{x\sqrt{1-x^2}}\,dx$

$$\frac{dI(a)}{da} = \int_0^1 \frac{x}{(1+a^2x^2)}\,\frac{1}{x\sqrt{1-x^2}}\,dx = \int_0^1 \frac{dx}{(1+a^2x^2)\sqrt{1-x^2}}$$

Put $x = \sin t \;\;\Rightarrow\;\; dx = \cos t\,dt$

L.L.: $x = 0 \;\;\Rightarrow\;\; t = 0$

U.L.: $x = 1 \;\;\Rightarrow\;\; t = \frac{\pi}{2}$

$$\frac{dI(a)}{da} = \int_0^{\pi/2} \frac{1}{1+a^2\sin^2 t}\,\frac{1}{\cos t}\cos t\,dt = \int_0^{\pi/2} \frac{dt}{1+a^2\sin^2 t}$$

$$= \int_0^{\pi/2} \frac{\sec^2 t\,dt}{1+(1+a^2)\tan^2 t} = \frac{1}{\sqrt{1+a^2}}\tan^{-1}\left(\sqrt{1+a^2}\tan t\right)\Big]_0^{\pi/2} = \frac{1}{\sqrt{1+a^2}}\cdot\frac{\pi}{2}$$

$$\Rightarrow\;\; I(a) = \frac{\pi}{2}\log_e\left(a+\sqrt{1+a^2}\right) + c$$

But $\;I(0) = 0 \;\Rightarrow\; c = 0$

$$\Rightarrow\;\; I(a) = \frac{\pi}{2}\log_e\left(a+\sqrt{1+a^2}\right).$$

Ex.21: If a real valued function f is given by $\int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x} + b,\; x > 0$, where $a > 0$ and b are real constants, then $f(4)$ is equal to

(a) 4 (b) 6 (c) 8 (d) 10 **[JAM-CA-2010]**

Soln. $\int_a^x \frac{f(t)}{t^2}\,dt = 2\sqrt{x} + b$

Taking derivative on both sides w.r.t. x

$\frac{f(x)}{x^2} = \frac{1}{\sqrt{x}} \;\Rightarrow\; f(x) = x^{3/2}$. So, $f(4) = 4^{3/2} = 8.$

Hence, correct option is (c).

Ex.22: Let $f(x) = \int_{\sin x}^{\cos x} e^{-t^2}\,dt$, then $f'\left(\frac{\pi}{4}\right)$ equals

(a) $\sqrt{\frac{1}{e}}$ (b) $-\sqrt{\frac{2}{e}}$ (c) $\sqrt{\frac{2}{e}}$ (d) $-\sqrt{\frac{1}{e}}$ **[JAM-MA-2006]**

Soln. Applying Leibnitz rule

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)\,dt = f(v(x))\,\frac{dv}{dx} - f(u(x))\,\frac{du}{dx}$$

$$f(x) = \int_{\sin x}^{\cos x} e^{-t^2} dt$$

$$f'(x) = e^{-\cos^2 x}(-\sin x) - e^{-\sin^2 x}(\cos x)$$

$$f'\left(\frac{\pi}{4}\right) = e^{-1/2} \cdot \left(-\frac{1}{\sqrt{2}}\right) - e^{-1/2} \cdot \left(\frac{1}{\sqrt{2}}\right) = -\sqrt{\frac{2}{e}}.$$

Hence, correct option is (b).

7. BETA AND GAMMA FUNCTION

7.1 BETA FUNCTION

Beta function is denoted by $B(l, m)$ and defined as $B(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1} dx$, where l, m are positive numbers.

7.1.1 Symmetric property of Beta function $B(l, m) = B(m, l)$

Proof : $B(l, m) = \int_0^1 x^{l-1}(1-x)^{m-1} dx = \int_0^1 (1-x)^{l-1}\{1-(1-x)\}^{m-1} dx = \int_0^1 x^{m-1}(1-x)^{l-1} dx = B(m, l).$

7.2 GAMMA FUNCTION

Gamma function denoted by $\Gamma n = \int_0^\infty e^{-x} x^{n-1} dx$, $n > 0$.

7.2.1 RELATION BETWEEN BETA FUNCTION AND GAMMA FUNCTION

$$B(l, m) = \frac{\Gamma l \, \Gamma m}{\Gamma(l+m)}$$

7.2.2 PROPERTIES OF GAMMA FUNCTION

(i) $\Gamma n = (n-1)\,\Gamma(n-1)$
If n is positive integer
$\Gamma n = (n-1)!$

(ii) $\Gamma 1 = 1$

(iii) $\Gamma 0 = \infty$

(iv) $\Gamma(-n) = \infty$ where n is positive integer

(v) $\Gamma \frac{1}{2} = \sqrt{\pi}$

(vi) $\left|\frac{1}{n}\right. \left|1-\frac{1}{n}\right. = \dfrac{\pi}{\sin \frac{\pi}{n}}$, n is a positive integer.

7.2.3 TRANSFORMATION OF GAMMA FUNCTION

(i) Put $x = \log\left(\frac{1}{y}\right)$

$$\Gamma n = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy$$

(ii) Put $x = cy$

$$\int_0^\infty e^{-cy} y^{n-1} dy = \frac{\Gamma n}{c^n}$$

(iii) Put $x^n = y$

$$\int_0^\infty e^{-y^{(1/n)}} dy = \Gamma(n+1)$$

Proof (i): We have

Let $x = \log\left(\frac{1}{y}\right) \Rightarrow dx = -\frac{1}{y}dy \Rightarrow \quad dy = -ydx$

limit at x, at $y \to 0, \quad x \to \infty$

$y \to 1, \quad x \to 0$

$$\therefore \quad \int_0^1 \log\left(\frac{1}{y}\right)^{n-1} dy = \int_\infty^0 -(x)^{n-1}\, ydx = \int_0^\infty x^{n-1}e^{-x}dx = \overline{|(n)} \text{ Proved.}$$

Similarly do proof (ii) and (iii)

7.2.4 <u>TRANSFORMATION OF BETA FUNCTION</u>

(i) $B(l,m) = \int_0^\infty \frac{y^{l-1}}{(1+y)^{l+m}}dy$

Since, $B(l,m) = B(m,l)$. So, $B(m,l) = \int_0^\infty \frac{y^{m-1}}{(1+y)^{l+m}}dy$

Proof : Put $x = \frac{1}{1+y}$ limit of x, $y \to 0, x \to 1$

$y \to \infty, x \to 0$

$dx = \frac{-dy}{(1+y)^2}$

$$\therefore \quad B(l,m) = \int_0^\infty \frac{y^{\ell-1}}{(1+y)^{\ell+m}}dy = \int_1^0 \left(\frac{1-x}{x}\right)^{\ell-1}(1+y)^2 x^{\ell+m}(-dx)$$

$$= \int_0^1 \frac{(1-x)^{\ell-1}x^{\ell+m-\ell+1}}{(x)^2}dx = \int_0^1 x^{m-1}(1-x)^{\ell-1}dx = \beta(m,\ell)$$

(ii) Put $x = \sin^2\theta$

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \quad \text{(generalized Walli's Formula)}$$

$p, q > -1$

Ex.23: $I = \int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x}\, dx$

Soln. $I = \int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}} \times \int_0^{\pi/2} \sqrt{\sin x}\, dx = \int_0^{\pi/2} \sin^{-1/2} x\cos^0 x\, dx \times \int_0^{\pi/2} \sin^{1/2} x\cos^0 x\, dx$

$$= \frac{\overline{\left|\frac{1}{4}\right.}\,\overline{\left|\frac{1}{2}\right.}}{2\overline{\left|\frac{3}{4}\right.}} \times \frac{\overline{\left|\frac{3}{4}\right.}\,\overline{\left|\frac{1}{2}\right.}}{2\overline{\left|\frac{5}{4}\right.}} = \frac{\overline{\left|\frac{1}{4}\right.}\,\overline{\left|\frac{1}{2}\right.}\,\overline{\left|\frac{1}{2}\right.}}{\frac{1}{4}\cdot\overline{\left|\frac{1}{4}\right.}} = \pi.$$

Ex.24: $I=\int_0^1 \frac{x^2dx}{(1-x^4)^{1/2}}\times\int_0^1\frac{dx}{(1+x^4)^{1/2}}$

Soln. $I=\int_0^1 \frac{x^2dx}{(1-x^4)^{1/2}}\times\int_0^1\frac{dx}{(1+x^4)^{1/2}}=I_1\times I_2$

$$I_1=\int_0^1\frac{x^2}{(1-x^4)^{1/2}}dx$$

Let us put $x^2=\sin\theta$; $2x\,dx=\cos\theta\,d\theta$

So, $I_1=\frac{1}{2}\int_0^{\pi/2}\frac{\sqrt{\sin\theta}}{\cos\theta}\cos\theta\,d\theta=\frac{1}{2}\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta$ and $I_2=\int_0^1\frac{dx}{(1+x^4)}\frac{1}{2}$

Let $x^2=\tan\phi$; $2x\,dx=\sec\phi\,d\phi$

So, $I_2=\frac{1}{2}\int_0^{\pi/4}\frac{\sec^2\phi\,d\phi}{\sqrt{\tan\phi\sec\phi}}=\frac{1}{\sqrt{2}}\int_0^{\pi/4}\frac{d\phi}{\sqrt{\sin 2\phi}}=\frac{1}{2\sqrt{2}}\int_0^{\pi/2}(\sin\alpha)^{-1/2}d\alpha$ Put $2\phi=\alpha$

So, $I=I_1I_2=\frac{1}{2}\int_0^{\pi/2}\sqrt{\sin\theta}\,d\theta\times\frac{1}{2\sqrt{2}}\int_0^{\pi/2}(\sin\alpha)^{-1/2}d\alpha=\frac{\pi}{4\sqrt{2}}.$

Ex.25: $\int_0^1\frac{dx}{(1-x^n)^{1/n}}$

Soln. $I=\int_0^1\frac{dx}{(1-x^n)^{1/n}}$

Let $x^n=\sin^2\theta$

i.e. $x=\sin^{\frac{2}{n}}\theta$

$$dx=\frac{2}{n}\sin^{\left(\frac{2}{n}-1\right)}\theta\cos\theta\,d\theta$$

So, $I=\frac{2}{n}\int_0^{\pi/2}\frac{\sin^{\left(\frac{2}{n}-1\right)}\theta\cos\theta\,d\theta}{\cos\theta}=\frac{2}{n}\int_0^{\pi/2}\sin^{\frac{2}{n}-1}\theta\,d\theta=\frac{2}{n}\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}=\frac{\sqrt{\pi}}{n}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{n+2}{2n}\right)}.$

Ex.26: $\int_a^b\frac{dx}{\sqrt{(x-a)(b-x)}}$

Soln. Let $x=a\cos^2\theta+b\sin^2\theta$

$dx=2a\cos\theta\sin\theta\,d\theta+2b\sin\theta\cos\theta\,d\theta=2(b-a)\sin\theta\cos\theta\,d\theta$

$x-a=a\cos^2\theta+b\sin^2\theta-a=(b-a)\sin^2\theta$

$b-x=b-a\cos^2\theta-b\sin^2\theta=(b-a)\cos^2\theta$

So, $I=\int_0^{\pi/2}\frac{2(b-a)\sin\theta\cos\theta}{(b-a)\sin\theta\cos\theta}=2\int_0^{\pi/2}d\theta=\pi.$

Ex.27: Show, by means of a suitable substitution, that $\int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta\, d\theta = \frac{1}{2}\int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}}\, dt,\ x, y > 0$.

Soln. Let $\sin\theta = \frac{1}{\sqrt{1+z}}$, then $\cos\theta = \frac{z^{1/2}}{\sqrt{1+z}}$

$$\cos\theta\, d\theta = -\frac{1}{2}(1+z)^{-3/2}\, dz$$

$$I = \int_0^{\pi/2} \sin^{2x-1}\theta \cos^{2y-1}\theta\, d\theta = -\frac{1}{2}\int_{\infty}^{0} \frac{1}{(1+z)^{x-1/2}} \cdot \frac{z^{y-1}}{(1+z)^{y-1}} \cdot \frac{1}{(1+z)^{3/2}}\, dz = \frac{1}{2}\int_0^{\infty} \frac{z^{y-1}}{(1+z)^{x+y}}\, dz = \frac{1}{2} B(y, x)$$

Since, $B(x, y) = B(y, x)$

So, $I = \frac{1}{2}\int_0^{\infty} \frac{z^{y-1}}{(1+z)^{x+y}}\, dz$.

8. DIFFERENTIATION OF DEFINITE INTEGRAL :

Leibnitz Rule : If the functions $\phi(x)$ and $\psi(x)$ are defined on $[a,b]$ and differentiable at a point $x \in (a,b)$ and $f(x,t)$ is continuous. Then

(i) $$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(x,t)\, dt\right] = \int_{\phi(x)}^{\psi(x)} \frac{\partial f}{\partial x} f(x,t)\, dt + \frac{d}{dx}\{\psi(x)\} \cdot f(x, \psi(x)) - \frac{d}{dx}\{\phi(x)\} \cdot f(x, \phi(x))$$

If f is a function of t only, then

(ii) $$\frac{d}{dx}\left[\int_{\phi(x)}^{\psi(x)} f(t)\, dt\right] = \frac{d}{dx}\{\psi(x)\} \cdot f(\psi(x)) - \frac{d}{dx}\{\phi(x)\} \cdot f(\phi(x))$$

Ex.29: Evaluate $\lim_{x\to\infty} \frac{\left(\int_0^x e^{t^2} dt\right)^2}{\int_0^x e^{2t^2} dt}$

Soln. $\lim_{x\to\infty} \frac{\left(\int_0^x e^{t^2} dt\right)^2}{\int_0^x e^{2t^2} dt}$ $\left(\frac{\infty}{\infty} \text{ form}\right)$

(Applying L' Hospital rule), $= \lim_{x\to\infty} \frac{2\left(\int_0^x e^{t^2} dt\right) \cdot e^{x^2}}{e^{2x^2}} = \lim_{x\to\infty} \frac{2\int_0^x e^{t^2} dt}{e^{x^2}}$

(Again L' Hospital rule), $= \lim_{x\to\infty} \frac{2e^{x^2}}{2x \cdot e^{x^2}} = 0$.

Ex.30: If $\{f(x)\}^{101} = \int_0^x (f(t))^{100} \cdot \frac{1}{1+\sin t}\, dt$, then find $f(x)$

Soln. $(f(x))^{101} = \int_0^x (f(t))^{100} \cdot \frac{1}{1+\sin t}\, dt$, differentiate w.r.t. x, gives $101(f(x))^{100} \cdot f'(x) = (f(x))^{100} \cdot \frac{1}{1+\sin x}$

$$\Rightarrow \left(f(x)\right)^{100}\left[101\, f'(x) - \frac{1}{1+\sin x}\right] = 0, \text{ either } \left(f(x)\right)^{100} = 0 \Rightarrow f(x) = 0 \text{ or } 101\ f'(x) - \frac{1}{1+\sin x} = 0$$

$$\Rightarrow f'(x) = \frac{1}{101(1+\sin x)} \Rightarrow f'(x) = \frac{1}{101}\left[\frac{1-\sin x}{\cos^2 x}\right] \Rightarrow f'(x) = \frac{1}{101}\left[\sec^2 x - \sec x \cdot \tan x\right]$$

$$\Rightarrow f(x) = \frac{1}{101}\left[\tan x - \sec x\right] + c$$

Since, $f(0) = 0$

$$\Rightarrow 0 = \frac{1}{101}[0-1] + c \Rightarrow c = \frac{1}{101}$$

$$\therefore \boxed{f(x) = \frac{1}{101}\left[\tan x - \sec x + 1\right]}$$

Ex.31: The sum of $B(m+1, n)$ and $B(m, n+1)$ is

(a) $B(m, n)$ (b) $B(m+1, n+1)$ (c) $B(2m+1, 2n+1)$ (d) $2B(m, n)$ **[B.H.U.-2014]**

Soln. $B(m+1,n) + B(m,n+1) = \int_0^1 x^{m+1-1}(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^{n+1-1}dx$

$$= \int_0^1 x^m(1-x)^{n-1} + x^{m-1}(1-x)^n\,dx \quad = \int_0^1 x^{m-1}(1-x)^{1-n}(x+1-x)dx$$

$$= \int_0^1 x^{m-1}(1-x)^{1-n}dx = B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Ex.32: The value of $\Gamma\left(\frac{7}{2}\right)$ is

(a) $\frac{15\sqrt{\pi}}{8}$ (b) $\frac{3\sqrt{\pi}}{4}$ (c) $\frac{3\pi}{4}$ (d) $\frac{15\pi}{8}$ **[B.H.U.-2014]**

Soln. $\Gamma(n) = (n-1)\Gamma(n-1)$

$$\Rightarrow \Gamma\left(\frac{7}{2}\right) = \left(\frac{7}{2}-1\right)\Gamma\left(\frac{7}{2}-1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\times\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{5}{2}\times\frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{15}{8}\sqrt{\pi}$$

Ex.33: If a function $f:[0,1]\to\mathbb{R}$ is defined by $f(x) = 2rx$, when $\frac{1}{r+1} < x < \frac{1}{r}$ and r is a positive integer, then $\int_0^1 f(x)\,dx$ is **[B.H.U.-2015]**

(a) $\frac{\pi}{6}$ (b) $\frac{\pi^2}{6}$ (c) $\frac{\pi}{3}$ (d) $\frac{\pi^2}{3}$

Soln. $\int_0^1 f(x)dx = \sum_{r=1}^{\infty}\int_{1/r+1}^{1/r} f(x)dx = \sum_{r=1}^{\infty}\int_{1/r+1}^{1/r} 2rx\,dx = \sum_{r=1}^{\infty} r\left(\frac{1}{r^2} - \frac{1}{(r+1)^2}\right) = \sum_{r=1}^{\infty}\frac{1}{r} - \frac{r}{(1+r)^2} = \frac{\pi^2}{6}$

Ex.34: Let f be a continuously differentiable function defined on an interval $[a,b]$ such that $f(a)=f(b)=0$ and $\int_a^b f^2(x)dx=1$. Then the value of $\int_a^b xf(x)f'(x)dx$ is **[CUCET-2016]**

(a) $-\frac{1}{2}$ (b) 0 (c) $\frac{1}{2}$ (d) 1

Soln. $\int_a^b xf(x)f'(x)dx=\left.\frac{xf(x)^2}{2}\right|_a^b-\frac{1}{2}\int_a^b f^2(x)dx$ **(Using by part)**

$$=\frac{b(f(b))^2-a(f(a))^2}{2}-\frac{1}{2}=-\frac{1}{2}$$

Ex.35: The value of $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right)\Gamma\left(\frac{3}{9}\right)...\Gamma\left(\frac{8}{9}\right)$ is **[ISM-2017]**

(a) $16\pi^4/3$ (b) $8\pi^3/9$ (c) $32\pi^5/27$ (d) $4\pi^2/3$

Soln. We have $\overline{|(1-n)}\overline{|(n)}=\frac{\pi}{\sin n\pi}$

$$\therefore \quad \overline{\left|\left(\frac{1}{9}\right)\right.}\overline{\left|\left(\frac{2}{9}\right)\right.}......\overline{\left|\left(\frac{8}{9}\right)\right.}=\left(\overline{\left|\left(\frac{1}{9}\right)\right.}\overline{\left|\left(1-\frac{1}{9}\right)\right.}\right)\left(\overline{\left|\left(\frac{2}{9}\right)\right.}\overline{\left|\left(1-\frac{2}{9}\right)\right.}\right)\left(\overline{\left|\left(\frac{3}{3}\right)\right.}\overline{\left|\left(1-\frac{3}{9}\right)\right.}\right)\left(\overline{\left|\left(\frac{4}{9}\right)\right.}\overline{\left|\left(1-\frac{4}{9}\right)\right.}\right)$$

$$=\frac{\pi}{\sin\frac{\pi}{9}}\times\frac{\pi}{\sin\frac{2\pi}{9}}\times\frac{\pi}{\sin\frac{3\pi}{9}}\times\frac{\pi}{\sin\frac{4\pi}{9}}=\frac{\pi^4}{\sin\frac{\pi}{9}\cdot\sin\frac{2\pi}{9}\cdot\sin\frac{3\pi}{9}\cdot\sin\frac{4\pi}{9}}=\frac{\pi^4}{\frac{\sqrt{3}}{2}\left(\sin\frac{\pi}{9}\cdot\sin\frac{2\pi}{9}\cdot\sin\frac{4\pi}{9}\right)}$$

$4\sin\theta\cdot\sin2\theta\cdot\sin4\theta=-\sin\theta+\sin3\theta+\sin5\theta-\sin7\theta=\frac{\sqrt{3}}{2}$ (By using complex number Euler formula)

$$\Rightarrow \quad \sin\theta\cdot\sin2\theta\cdot\sin4\theta=\frac{\sqrt{3}}{8}$$

$$\therefore \quad \overline{\left|\left(\frac{1}{9}\right)\right.}..........\overline{\left|\left(\frac{8}{9}\right)\right.}=\frac{16\pi^4}{3}$$

Ex.36: The value of integral $\int_0^\infty e^{-x^2}\cos\alpha x dx$ is **[ISM-2017]**

(a) $\frac{1}{4}\sqrt{\pi}e^{-\frac{1}{4}\alpha^2}$ (b) $\frac{1}{2}\sqrt{\pi}e^{-\frac{1}{4}\alpha^2}$ (c) $\frac{1}{2}\pi e^{-\frac{1}{4}\alpha^2}$ (d) $\frac{1}{2}\sqrt{\pi}e^{-\frac{1}{2}\alpha^2}$

Soln. $\int_0^\infty e^{-x^2}\cos\alpha x dx=\int_0^\infty e^{-x^2}\frac{e^{i\alpha x}+e^{-i\alpha x}}{2}dx$

$$=\frac{1}{2}\int_0^\infty(e^{-x^2+i\alpha x}+e^{-x^2-\alpha i x})dx=\frac{1}{2}\int_0^\infty e^{-\left(x^2+i\alpha x-\frac{\alpha^2}{4}+\frac{\alpha^2}{4}\right)}+e^{-\left(x^2+\alpha i x-\frac{\alpha^2}{4}+\frac{\alpha^2}{4}\right)}dx$$

$$= \frac{1}{2} e^{-\alpha^2/4} \int_0^\infty e^{-\left(x-\frac{i\alpha}{2}\right)^2} + e^{-\left(x+\frac{\alpha i}{2}\right)^2} dx = \frac{1}{2} e^{-\alpha^2/4}\left[\int_0^\infty e^{-\left(x-\frac{i\alpha}{2}\right)^2} dx + e^{-\left(x+\frac{\alpha i}{2}\right)^2} dx\right]$$

$$= \frac{1}{2} e^{-\alpha^2/4}\left[\int_{-i\alpha/2}^{\infty} e^{-t^2}dt + \int_{i\alpha/2}^{\infty} e^{-t^2}dt\right] = \frac{1}{2} e^{-\alpha^2/4}\left[\int_{-\infty}^{i\alpha/2} e^{-t^2}dt + \int_{i\alpha/2}^{\infty} e^{-t^2}dt\right]$$

$$= \frac{1}{2} e^{-\alpha^2/4} \times 2\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}e^{-\alpha^2/4}}{2}$$

Note: $\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$ **(Expain in next chapter)**

Ex.37: The value of the integral $\int_0^{10} (x-[x])dx$ is **[HCU-2012]**

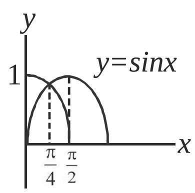
(a) 2 (b) 3 (c) 4 (d) 5

Soln. $\int_0^{10} (x-[x])dx = \int_0^{10} \{x\}dx = \int_0^{1\cdot 10} \{x\}dx = 10\int_0^1 \{x\}dx = 10\int_0^1 x dx = 5$

Ex.38: The integral $\int_0^{\pi/2} \min(\sin x, \cos x)dx$ equals **[JAM(CA)-2007]**

(a) $\sqrt{2}-2$ (b) $2-\sqrt{2}$ (c) $2\sqrt{2}$ (d) $2+\sqrt{2}$

Soln. $\therefore \quad \min(\sin x, \cos x) = \begin{cases} \sin x, & 0 \le x \le \frac{\pi}{4} \\ \cos x, & \frac{\pi}{4} \le x \le \frac{\pi}{2} \end{cases}$



$$\therefore \quad \int_0^{\pi/2} \min(\sin x, \cos x)dx = \int_0^{\pi/4} \sin x dx + \int_{\pi/4}^{\pi/2} \cos x dx = 1-\frac{1}{\sqrt{2}}+1-\frac{1}{\sqrt{2}} = 2-\sqrt{2}$$

Ex.39: The value of the limit **[JAM(MS)-2014]**

$$\lim_{x\to\frac{1}{2}} \frac{\int_{1/2}^{x} \cos^2 \pi t dt}{\frac{e^{2x}}{2} - e\left(x^2+\frac{1}{4}\right)} \text{ is}$$

(a) 0 (b) $\frac{\pi}{e}$ (c) $\frac{\pi^2}{2e}$ (d) $-\frac{\pi^2}{2e}$

Soln. $\lim_{x\to\frac{1}{2}} \frac{\int_{1/2}^{x} \cos^2 \pi t dt}{\frac{e^{2x}}{2} - e\left(x^2+\frac{1}{4}\right)}$

By L′ Hospital rule

$$\lim_{x\to\frac{1}{2}} \frac{\cos^2 \pi x}{e^{2x} - 2ex}$$

again by L' Hospital rule

$$= \lim_{x\to\frac{1}{2}} \frac{-\pi\sin 2\pi x}{2e^{2x} - 2e} = \lim_{x\to\frac{1}{2}} \frac{-2\pi^2 \cos 2\pi x}{4e^{2x}} = \frac{2\pi^2}{4e} = \frac{\pi^2}{2e}$$

Ex.40: The length of the curve $y = \sqrt{4 - x^2}$ from $x = -\sqrt{2}$ to $x = \sqrt{2}$ is equal to __________

[JAM(MS)-2015]

Soln. We have,

$$y = \sqrt{4 - x^2} \Rightarrow dy = \frac{-x}{\sqrt{4 - x^2}}$$

$$\text{Length} = \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_{-\sqrt{2}}^{\sqrt{2}} \sqrt{1 + \frac{x^2}{4 - x^2}}\, dx$$

$$= 2\int_0^{\sqrt{2}} \sqrt{\frac{4}{4 - x^2}}\, dx = 4\int_0^{\sqrt{2}} \frac{1}{\sqrt{4 - x^2}}\, dx = 4\sin^{-1}\left(\frac{x}{2}\right)\Bigg|_0^{\sqrt{2}} = 4 \times \frac{\pi}{4} = \pi$$

Ex.41: Let $f : [0, \infty) \to [0, \infty)$ be a twice differentiable and increasing function with $f(0) = 0$. Suppose that, for any $t \geq 0$, the length of the arc of the curve $y = f(x), x \geq 0$ between $x = 0$ and $x = t$ is $\frac{2}{3}\left[(1+t)^{3/2} - 1\right]$. Then $f(4)$ is equal to **[JAM(MS)-2013]**

(a) $\frac{11}{3}$ (b) $\frac{13}{3}$ (c) $\frac{14}{3}$ (d) $\frac{16}{3}$

Soln. Length $= \int_0^t \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \frac{2}{3}\left[(1+t)^{3/2} - 1\right]$ (Given)

Differentiating w.r.t. t we get,

$$\sqrt{1 + \left(f'(t)\right)^2} = \frac{2}{3} \times \frac{3}{2}(1+t)^{1/2}$$

$$\Rightarrow\ 1 + (f'(t))^2 = 1 + t \Rightarrow (f'(t))^2 = t \Rightarrow f'(t) = \sqrt{t} \Rightarrow f(t) = \frac{2}{3}x^{3/2} + c \qquad (f'(t) \geq 0 \text{ Given})$$

$$f(0) = 0 \Rightarrow c = 0$$

$$\therefore\quad f(x) = \frac{2}{3}x^{3/2}$$

$$\therefore\quad f(4) = \frac{16}{3}$$

MISCELLANEOUS

Ex.1: Which Integral is greater ?

$$I_1 = \int_0^{\pi/2} \sin^{10} x\, dx,\ I_2 = \int_0^{\pi/2} \sin^5 x\, dx,$$

Soln. $\therefore\ \sin^5 x > \sin^{10} x\ \forall x \in \left(0, \frac{\pi}{2}\right)$

$$\Rightarrow \int_0^{\pi/2} \sin^5 x\, dx > \int_0^{\pi/2} \sin^{10} x\, dx$$

Hence, $I_1 > I_2$

Ex.2: If $f(\pi) = 2$ and $\int_0^{\pi} \left(f(x) + f''(x)\right)\sin x dx = 5$. Then the value of $f(0)$ is

(a) 3 (b) 2 (c) 0 (d) 6

Soln. Given,

$f(\pi) = 2$

$$\therefore \int_0^{\pi} \left(f(x) + f''(x)\right)\sin x \cdot dx = 5$$

$$\Rightarrow \int_0^{\pi} \underset{\mathrm{I}}{f(x)} \underset{\mathrm{II}}{\sin x} dx + \int_0^{\pi} f''(x)\sin x\, dx = 5$$

$$\Rightarrow \left[f(x)(-\cos x)\right]_0^{\pi} + \int_0^{\pi} \underset{\mathrm{I}}{f'(x)} \underset{\mathrm{II}}{\cos x}\, dx + \int_0^{\pi} f''(x)\cdot \sin x\, dx = 5$$

$$\Rightarrow\ f(\pi) + f(0) + \left[f'(x)\sin x\right]_0^{\pi} - \int_0^{\pi} f''(x)\cdot \sin x\, dx + \int_0^{\pi} f''(x)\sin x\, dx = 5$$

$$\Rightarrow f(\pi) + f(0) = 5 \qquad \text{...(i)}$$

$f(\pi) = 2$

$\Rightarrow$ from (*), we get

$\Rightarrow 2 + f(0) = 5$

$\Rightarrow \boxed{f(0) = 3}$

Ex.3: Let $g(x)$ is the inverse of $f(x)$ and $f(x)$ has domain $x \in [1,5]$, where $f(1) = 2$ and $f(5) = 10$. Then the value of $\int_1^5 f(x)dx + \int_2^{10} g(y)dy$ is

(a) 48 (b) 50 (c) 7 (d) 20

Soln. Given,

$g(x)$ is the inverse of $f(x)$

$\Rightarrow f^{-1}(x) = g(x)$ and $g^{-1}(x) = f(x)$

$\therefore\ f(1) = 2$ and $f(5) = 10$

$\Rightarrow g(2)=1$ and $g(10)=5$

$\therefore \int_1^5 f(x)dx+\int_2^{10} g(y)dy$ ....(i)

Consider,

$\int_2^{10} g(y)dy$ ($\because$ when $y = 2, t = 1,$

$y = 10, t = 5)$

Let $g(y)=t$ ($\because g^{-1}(x)=f(x)$)

$y=g^{-1}(t)$

$y=f(t)$

$dy=f'(t)dt$

$\therefore \int_1^5 t\cdot f'(t)dt$

I II

$=\left[t\ f(t)\right]_1^5-\int_1^5 f(t)dt$

$=5f(5)-f(1)-\int_1^5 f(x)dx$ $\left(\because \int_a^b f(t)dt=\int_a^b f(x)dx\right)$

$=50-2-\int_1^5 f(x)dx$

$=48-\int_1^5 f(x)dx$

From (1), we get

$\therefore \int_1^5 f(x)\,dx+\int_2^{10} g(y)dy$

$=\int_1^5 f(x)dx+48-\int_1^5 f(x)dx=48$

Correct option is (a)

Ex.4: The value of integral $\int_0^{12}\left[\sqrt{x}\right]dx$, where $[\cdot]$ denotes the greatest Integer function

(a) 24 (b) 22 (c) $\dfrac{1}{2}$ (d) 10

Soln. $\int_0^{12} \left[\sqrt{x}\right] dx$,

Let $\sqrt{x} = I \Rightarrow x = I^2$

So, we can break the Integral

0 1 4 9 12

$$\therefore \int_0^{12} \left[\sqrt{x}\right] dx = \int_0^1 0\,dx + \int_1^4 1\,dx + \int_4^9 2\,dx + \int_9^{12} 3\,dx$$

$= 3 + 10 + 9 = 22$

Hence, correct option is (b)

Ex. 5: The value of Integral $\int_{-1}^{3} \left\{ \tan^{-1}\left(\frac{x}{x^2+1}\right) + \tan^{-1}\left(\frac{x^2+1}{x}\right) dx \right\}$ is

(a) 0 (b) $\frac{\pi}{2}$ (c) π (d) 2π

Soln. Let $I = \int_{-1}^{3} \left\{ \tan^{-1}\left(\frac{x}{x^2+1}\right) + \tan^{-1}\left(\frac{x^2+1}{x}\right) \right\} dx$

–1 1 3

$$I = \int_{-1}^{1} \left\{ \tan^{-1}\left(\frac{x}{x^2+1}\right) + \tan^{-1}\left(\frac{x^2+1}{x}\right) \right\} dx + \int_{-1}^{3} \left\{ \tan^{-1}\left(\frac{x}{x^2+1}\right) + \tan^{-1}\left(\frac{1}{\frac{x}{x^2+1}}\right) \right\} dx$$

$$I = 0 + \int_{1}^{3} \left\{ \tan^{-1}\left(\frac{x}{x^2+1}\right) + \tan^{-1}\left(\frac{1}{\frac{x}{x^2+1}}\right) \right\} dx$$

$$I = \int_{1}^{3} \left\{ \tan^{-1}\left(\frac{x}{x^2+1}\right) + \tan^{-1}\left(\frac{1}{\frac{x}{x^2+1}}\right) \right\} dx$$

$$\because \tan^{-1}\left(\frac{1}{a}\right) = \begin{cases} \cot^{-1} a & , a > 0 \\ -\pi + \cot^{-1}(a) & , a < 0 \end{cases}$$

from (i), we get

$$I = \int_{1}^{3} \left\{ \tan^{-1}\left(\frac{x}{x^2+1}\right) + \cot^{-1}\left(\frac{x}{x^2+1}\right) \right\} dx$$

$= \int_1^3 \frac{\pi}{2} dx$ $\qquad (\because \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \ \forall x \in \mathbb{R})$

$= \pi$

Correct option is (c)

Ex.6: Let $G(x) = \int_2^{x^2} \frac{dt}{1+\sqrt{t}} (x > 0)$. Then the value of $G'(9)$ is

(a) $\frac{9}{5}$ (b) $\frac{2}{5}$ (c) 0 (d) $\frac{1}{3}$

Soln. Given

$$G(x) = \int_2^{x^2} \frac{1}{1+\sqrt{t}} dt, x > 0$$

By Leibnitz Rule, we get

$$G'(x) = \frac{1}{1+\sqrt{x^2}} \cdot 2x - 0$$

$$G'(x) = \frac{2x}{1+\sqrt{x^2}}$$

$$G'(9) = \frac{18}{10} = \frac{9}{5}$$

Correct option is (b)

Ex.7: Let $f(x) = \int_0^x e^{x-y} f'(y) dy - (x^2 - x + 1) e^x$. Then the value of $f\left(\frac{1}{2}\right)$ is

(a) 0 (b) 3 (c) $\frac{1}{2}$ (d) 7

Soln. Given

$$f(x) = \int_0^x e^{x-y} f'(y) dy - (x^2 - x + 1) \cdot e^x$$

$$f(x) = e^x \int_0^x e^{-y} \cdot f'(y) dy - e^x (x^2 - x + 1)$$

Multiply both sides e^{-x}, we get

$$e^{-x} \cdot f(x) = \int_0^x e^{-y} \cdot f'(y) dy - (x^2 - x + 1)$$

Differentiate both sides, we get

$$\Rightarrow e^{-x} f'(x) + f(x)(-e^{-x}) = e^{-x} \cdot f'(x) - (2x - 1)$$

$$\Rightarrow f(x) = e^x \cdot (2x - 1)$$

$$\therefore\ f\left(\frac{1}{2}\right)=0$$

Correct option is (c)

Ex.8: $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\int_{2}^{\sec^2 x} f(t)\,dt}{x^2-\frac{\pi^2}{16}}$ equals

(a) $\frac{8}{\pi}f(2)$ (b) $\frac{2}{\pi}f(2)$ (c) $\frac{2}{\pi}f\left(\frac{1}{2}\right)$ (d) $4f(2)$

Soln. $\lim\limits_{x\to\frac{\pi}{4}} \dfrac{\int_{2}^{\sec^2 x} f(t)\,dt}{x^2-\frac{\pi^2}{16}} \quad \left(\frac{0}{0}\right)$

By L Hopital's Rule, we get

$$\lim_{x\to\frac{\pi}{4}} \frac{\frac{d}{dx}\int_{2}^{\sec^2 x} f(t)\,dt}{2x}$$

$$\lim_{x\to\frac{\pi}{4}} \frac{f\left(\sec^2 x\right)\cdot 2\sec x\cdot\sec x\cdot\tan x-0}{2x}$$

$$=\frac{f(2)\cdot 2\cdot\sqrt{2}.\sqrt{2}}{2\times\frac{\pi}{4}}=\frac{8}{\pi}f(2)$$

Correct option is (a)

Ex.9: Let $I=\frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\frac{1}{\left(1+e^{\sin x}\right)(2-\cos 2x)}dx$ then the value of $27I^2$ is

(a) 3 (b) $\frac{7}{2}$ (c) 4 (d) 20

Soln. $I=\frac{2}{\pi}\int_{\pi/4}^{\pi/4}\frac{1}{\left(1+e^{\sin x}\right)(2-\cos 2x)}dx$

$$\because \int_{-a}^{a} f(x)\,dx=\int_{0}^{a}\left(f(x)+f(-x)\right)dx$$

$$I=\frac{2}{\pi}\int_{0}^{\pi/4}\left[\frac{1}{\left(1+e^{\sin x}\right)(2-\cos 2x)}+\frac{1}{\left(1+e^{-\sin x}\right)(2-\cos 2x)}\right]dx$$

$$I = \frac{2}{\pi}\int_0^{\pi/4} \frac{1+e^{\sin x}}{\left(1+e^{\sin x}\right)(2-\cos 2x)} dx$$

$$I = \frac{2}{\pi}\int_0^{\pi/4} \frac{1}{(2-\cos 2x)} dx$$

$$\therefore \cos 2x = \frac{1-\tan^2 x}{1+\tan^2 x} \qquad \textbf{...(*)}$$

form (*), we get

$$I = \frac{2}{\pi}\int_0^{\pi/4} \frac{1}{2-\left(\frac{1-\tan^2 x}{1+\tan^2 x}\right)} dx$$

$$I = \frac{2}{\pi}\int_0^{\pi/4} \frac{\sec^2 x}{3\tan^2 x+1} dx$$

let $t = \tan x$, when $x = \frac{\pi}{4}, t = 1$

$dt = \sec^2 x$ $\qquad x = 0,\ t = 0$

$$I = \frac{2}{\pi}\int_0^{1} \frac{1}{3t^2+1} dt$$

$$= \frac{2}{\pi}\int_0^{1} \frac{1}{\left(\sqrt{3}\,t\right)^2+(1)^2} dt$$

$$= \frac{2}{\pi}\cdot\frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{\sqrt{3}t}{1}\right)\right]_0^1$$

$$= \frac{2}{3\sqrt{3}}$$

$$\therefore 27\, I^2 = 27 \times \frac{4}{27} = 4$$

Correct option is (c)

Ex.10: The vlaue of integral $\int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5} d\theta$ is

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{3}$ (d) $\frac{3}{4}$

Soln. Let $\int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)} d\theta$

$$\therefore \int_0^{2a} f(x)\,dx = \int_0^{a} f(x) + f(2a-x)\,dx$$

$$I = \int_0^{\pi/4}\left(\frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5} + \frac{3\sqrt{\sin\theta}}{\left(\sqrt{\sin\theta}+\sqrt{\cos\theta}\right)^5}\right)d\theta$$

$$I = 3\int_0^{\pi/4}\left(\frac{\sqrt{\cos\theta}+\sqrt{\sin\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5}\right)d\theta$$

$$I = 3\int_0^{\pi/4}\frac{1}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^4}\,d\theta$$

Divide in Numerator and Denominator by $\left(\sqrt{\cos\theta}\right)^4$, we get

$$I = 3\int_0^{\pi/4}\frac{\sec^2\theta}{\left(1+\sqrt{\tan\theta}\right)^4}\,d\theta$$

Let $t = \sqrt{\tan\theta}$, when $\theta = \frac{\pi}{4}, t = 1$

$t^2 = \tan\theta$ $\qquad\qquad$ $\theta = 0, . t = 0$

$2t\,dt = \sec^2\theta\,d\theta$

$$I = 3\int_0^1 \frac{2t}{(1+t)^4}\,dt$$

$$I = 6\int_0^1 \frac{t}{(1+t)^4}\,dt$$

$$I = 6\int_0^1 \frac{t+1-1}{(1+t)^4}\,dt$$

$$I = 6\int_0^1 \left[\frac{(t+1)}{(1+t)^4} - \frac{1}{(1+t)^4}\right]dt$$

$$I = 6\int_0^1 \left[\frac{1}{(1+t)^3} - \frac{1}{(1+t)^4}\right]dt$$

$$= 6\left[\frac{1}{-2(1+t)^2} + \frac{1}{3}\cdot\frac{1}{(1+t)^3}\right]_0^1 = \frac{1}{2}$$

Correct option is (a)

Ex.11: Let $f(x)=\sin x+\int_0^x f'(t)\cdot\left\{2\sin t-\sin^2 t\right\}dt$. Then the value of $f\left(\frac{\pi}{6}\right)$ is

(a) 2 (b) 1 (c) 3 (d) $\frac{1}{3}$

Soln. $f(x)=\sin x+\int_0^x f'(t)\left\{2\sin t-\sin^2 t\right\}dt$ ...(i)

$\because\ f(x)=\sin x+\int_0^x f'(t)\cdot\left\{2\sin t-\sin^2 t\right\}dt$

By Leibnitz rule, we get

$f'(x)=\cos x+f'(x)\left(2\sin x-\sin^2 x\right)$

$f'(x)=\dfrac{\cos x}{1-2\sin x+\sin^2 x}$

$f'(x)=\dfrac{\cos x}{(\sin x-1)^2}$

Integrating both sides, we get

$f(x)=\int\dfrac{\cos x}{(\sin x-1)^2}dx$

Let $y=\sin x-1$

$dy=\cos x\,dx$

$\therefore\ f(x)=\int\dfrac{1}{y^2}dy$

$f(x)=-\dfrac{1}{y}+c$

$f(x)=-\dfrac{1}{\sin x-1}+C$ ...(*)

from (1), put $x=0$, we get

$\therefore\ f(0)=0$

From (*)

$\because f(0)=0$

$\therefore\ 0=1+C\Rightarrow\boxed{C=-1}$

$\because\ f(x)=-\dfrac{1}{(\sin x-1)}-1$

$f\left(\dfrac{\pi}{6}\right)=\dfrac{-1}{\frac{-1}{2}}-1=1$

Correct option is (b)

Ex.12: Let $I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx$ and $J = \int_0^1 \frac{\cos x}{\sqrt{x}} dx$ .Then, which one of the folllowing is True ?

(a) $I > \frac{2}{3}$ and $J > 2$

(b) $I < \frac{2}{3}$ and $J < 2$

(c) $I < \frac{2}{3}$ and $J > 2$

(d) $I > \frac{2}{3}$ and $J < 2$

Soln. Given,

$$I = \int_0^1 \frac{\sin x}{\sqrt{x}} dx$$

$\because\ x \in (0,1)$

We know that,

$\sin x < x\ \forall x > 0$

$$\frac{\sin x}{\sqrt{x}} < \frac{x}{\sqrt{x}}$$

$$\int_0^1 \frac{\sin x}{\sqrt{x}} dx < \int_0^1 \frac{x}{\sqrt{x}} dx$$

$$I < \left.\frac{x^{3/2}}{3/2}\right|_0^1$$

$$\boxed{I < \frac{2}{3}}$$

$\therefore$ We know that,

$\cos x < 1\ \forall x \in (0,1)$

$$\frac{\cos x}{\sqrt{x}} < \frac{1}{\sqrt{x}}$$

Integrating, we get

$$\int_0^1 \frac{\cos x}{\sqrt{x}} dx < \int_0^1 \frac{1}{\sqrt{x}} dx$$

$$J < \left. 2x^{1/2}\right|_0^1$$

$$J < 2$$

$\therefore\ \ I < \frac{2}{3}$ and $J < 2$

Correct option is (b)

Ex.13: Prove that, $0 < \int_0^1 \frac{x^7}{\left(1+x^8\right)^{1/3}} dx < \frac{1}{8}$

Soln. $\therefore\ x \in (0,1)$

$$0 < \frac{x^7}{\left(1+x^8\right)^{1/3}} < x^7\ \forall x \in (0,1)$$

Integrating, we get

$$\int_0^1 0\, dx < \int_0^1 \frac{x^7}{\left(1+x^8\right)^{1/3}} dx < \int_0^1 x^7 dx$$

$$0 < \int_0^1 \frac{x^7}{\left(1+x^8\right)^{1/3}} dx < \frac{1}{8}.$$

Ex.14: Let $f(x)$ is differentiable function satisfying $f(x) = \int_0^x e^t \cdot \sin(x-t)\, dt$ and $g(x) = f''(x) - f(x)$. Then the range of $g(x)$ is

(a) $\left[-\sqrt{2}, \sqrt{2}\right]$ (b) $\left(-\sqrt{3}, \sqrt{3}\right)$ (c) $(0, +\infty)$ (d) $\mathbb{R}$

Soln. Given,

$$f(x) = \int_0^x e^t \cdot \sin(x-t)\, dt$$

$$f(x) = \int_0^x e^{(x-t)} \cdot \sin(x-(x-t))\, dt \quad \left(\because \int_a^b f(x)\, dx = \int_a^b f(a+b-x)\, dx\right)$$

$$f(x) = \int_0^x e^{x-t} \sin t\, dt$$

$$f(x) = e^x \int_0^x e^{-t} \cdot \sin t\, dt$$

By Leibniz rule, we get

$$f'(x) = e^x \cdot e^{-x} \sin x + e^x \int_0^x e^{-t} \sin t\, dt$$

$$f'(x) = \sin x + e^x \int_0^x e^{-t} \sin t\, dt$$

Again differentiate both sides, we get

$$f''(x) = \cos x + e^x \cdot e^{-x} \sin x + e^x \int_0^x e^{-t} \sin t\, dt$$

$f''(x) = \cos x + \sin x + f(x)$

$\therefore\ g(x) = f''(x) - f(x)$

$= \cos x + \sin x + f(x) - f(x)$

$g(x) = \cos x + \sin x$

hence, range of $g(x)$ is $\left[-\sqrt{2}, \sqrt{2}\right]$

Correct option is (a)

Ex.15: Let $f(x)$ be a differentiable function with $f(0) = 0$ and $f'(x) + f(x) \le 1\ \forall x$ greater than (or) equal to 0. Then values not in range of $f(x)$ is/are

(a) 1 (b) 2 (c) 3 (d) 4

Soln. We know that

$\int e^x \left(f(x) + f'(x) \right) dx = e^x + c$

$\therefore\ f'(x) + f(x) \le 1$

multiply both sides by e^x, we get

$\Rightarrow e^x \left(f'(x) + f(x) \right) \le e^x$

$\Rightarrow \int_0^t e^x \left(f'(x) + f(x) \right) dt \le \int_0^t e^x dt$

$\Rightarrow e^x f(x) \Big|_0^t \le e^x \Big|_0^t \qquad (\because\ t \ge 0)$

$\Rightarrow e^t f(t) - e^o f(0) \le e^t - 1$

$\because\ f(0) = 0$

$e^t f(t) \le e^t - 1$

$f(t) \le 1 - e^{-t}$

So, range of $\left(1 - e^{-t}\right)$ is $(-\infty, 1)$

$f(t) \le$ range of function $\left(1 - e^{-t}\right)$ is $(-\infty, 1)$

$\Rightarrow f(t) < 1$

Option (a), (b), (c) and (d) are correct

Ex.16. The sum of the series $\sum_{n=0}^{\infty} \left(\frac{1}{3n+1} - \frac{1}{3n+2} \right)$ is

(a) $\frac{\pi}{3\sqrt{3}}$ (b) $\frac{\pi^2}{4}$ (c) $\frac{\pi}{2\sqrt{2}}$ (d) $\frac{\pi}{3\sqrt{7}}$

Soln. $\sum_{n=0}^{\infty} \left(\frac{1}{3n+1} - \frac{1}{3n+2} \right)$

If you observe that,

$$\frac{1}{3n+1}=\int_0^1 x^{3n}dx \text{ and } \frac{1}{3n+2}=\int_0^1 x^{3n+1}dx$$

$$\therefore \sum_{n=0}^{\infty}\left(\int_0^1 x^{3n}-x^{3n+1}dx\right)$$

$$\int_0^1\sum_{n=0}^{\infty}\left(\left(x^{3n}-x^{3n+1}\right)\right)dx$$

$$=\int_0^1\left(\sum_{n=0}^{\infty}x^{3n}(1-x)\right)dx$$

$$=\int_0^1(1-x)\sum_{n=0}^{\infty}x^{3n}dx$$

$\infty - G.P.$

$$=\int_0^1(1-x)\left(1+x^3+x^6+....+\infty\right)dx$$

$$=\int_0^1(1-x)\cdot\frac{1}{\left(1-x^3\right)}dx$$

$$=\int_0^1(1-x)\cdot\frac{1}{\left(1-x^3\right)}dx$$

$$=\int_0^1(1-x)\cdot\frac{1}{(1-x)\left(1+x+x^2\right)}dx$$

$$=\int_0^1\frac{1}{\left(1+x+x^2\right)}dx=\int_0^1\frac{1}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}dx$$

$$=\frac{1}{\frac{\sqrt{3}}{2}}\left[\tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right)\right]_0^1$$

$$=\frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{3}{\sqrt{3}}\right)-\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right]$$

$$=\frac{2}{\sqrt{3}}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{\pi}{3\sqrt{3}}$$

Ex.17: $I=\sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx,$ Then

(a) $I>\log_e 99$ (b) $I<\log_e 99$ (c) $I<\frac{49}{50}$ (d) $I>\frac{49}{50}$

Soln. $I=\sum_{k=1}^{98}\int_k^{k+1}\frac{k+1}{x(x+1)}dx$

$$\because\ x\in(k,k+1)$$

$x<k+1$ $\qquad$ $x>k$

$1<\frac{k+1}{x}$ $\qquad$ $1+x>1+k$

$\frac{1}{(1+x)}<\frac{(k+1)}{x(x+1)}$ ...(2) $\qquad$ $\frac{1}{x}>\frac{(1+k)}{x.(1+x)}$ ...(1)

From (1) and (2), we get

$$\int_k^{k+1}\frac{1}{(1+x)}dx<\int_k^{(k+1)}\frac{(k+1)}{x.(x+1)}dx<\int_k^{k+1}\frac{1}{x}dx$$

$$\log(k+2)-\log(k+1)<\int_k^{(k+1)}\frac{(k+1)}{x\cdot(x+1)}<\log(k+1)-\log k$$

$$\sum_{k=1}^{98}\log(k+2)-\log(k+1)<\sum_{k=1}^{98}\int_k^{k+1}\frac{(k+1)}{x(x+1)}dx<\sum_{k=1}^{98}\log(k+1)-\log k$$

$$\Rightarrow(\log 3-\log 2)+(\log 4-\log 3)+\ldots+(\log 100-\log 99)<I$$

$$<(\log 2-\log 1)+(\log 3-\log 2)+\ldots+(\log 99-\log 98)$$

$\log 100-\log 2<I<\log 99-\log 1$

$\log 50<I<\log 99$

$3.91<I<\log_e 99$

Correct option is (b) and (d)

Ex.18: If $(f(x))^{2007}=\int_0^x\frac{(f(t))^{2006}}{2+t^2}dt$. Then $f(x)$ is

(a) $\frac{1}{2007\cdot\sqrt{2}}\cdot\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$ (b) $\frac{1}{\sqrt{2}}\cdot\cot^{-1}\left(\frac{x}{\sqrt{3}}\right)$

(c) $\frac{1}{3\sqrt{2}}\cdot\tan^{-1}\left(\frac{x}{3\sqrt{3}}\right)$ (d) $\frac{1}{2006}\cdot\tan^{-1}\left(\frac{\sqrt{3}x}{2}\right)$

Soln. $\left(f(x)\right)^{2007} = \int_0^x \frac{\left(f(t)\right)^{2006}}{2+t^2}\,dt$ ...(i)

put $x=0$, we get

$\therefore\ f(0)=0$

from (1), we get

$$(2007)\left(f(x)\right)^{2006}\cdot f'(x) = \frac{f(x)^{2006}}{2+x^2}$$

$$\left(f(x)\right)^{2006}\left\{2007\cdot f'(x) - \frac{1}{2+x^2}\right\} = 0$$

$$\left(f(x)\right)^{2006} = 0$$

$$\Rightarrow f(x)=0$$

$$2007.\ f'(x) = \frac{1}{2+x^2}$$

Integerating, we get

$$2007\cdot f(x) = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)+c$$

$$\therefore\ f(0)=0 \Rightarrow \boxed{c=0}$$

$$2007 f(x) = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$$

$$\therefore\ f(x) = \frac{1}{2007\cdot\sqrt{2}}\cdot\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)$$

Hence, correct option is (a)

EXERCISE-1

1. Evaluate $\int_0^{100} \{x\}dx$, where $\{x\}$ denotes the fraction part of x.

2. Let $f:\mathbb{R}\to\mathbb{R}$ be defined as

$$f(t)=\begin{cases}\dfrac{\tan t}{t} & : \ t\neq 0\\ 1 & : \ t=0\end{cases}$$

Then find the value of $\lim\limits_{x\to 0}\dfrac{1}{x^2}\int_{x^2}^{x^3} f(t)\,dt$.

3. Find the interval in which $f(x)=\int_{-1}^{x}(e^t-1)(2-t)\,dt,\ x>-1$, is incresing.

4. If $f(t)=\int_2^3 \sin(x+t^2)\,dx$, find $f'(t)$.

5. If $f(x)=e^{g(x)}$ and $g(x)=\int_2^x \dfrac{t}{1+t^4}dt$, then find the value of $f'(2)$.

6. If $f(x)=\int_x^{x^2} x^2\sin t\,dt$, then find $f'(x)$.

7. Find the value of the function

$f(x)=1+x+\int_1^x\left((\ell n\, t)^2+2\,\ell n\, t\right)dt$, where $f'(x)$ vanishes.

8. If $f(x)=\begin{cases}0 & ; \ \text{when } x=\dfrac{n}{n+1},\ n=1,2,3,....\\ 1 & ; \ \text{otherwise}\end{cases}$

then the value of $\int_0^2 f(x)\,dx$

(a) 1 (b) 0 (c) 2 (d) ∞

9. Let $f:\mathbb{R}\to\mathbb{R}$ be a function with continuous derivative such that $f(\sqrt{2})=2$ and

$f(x)=\lim\limits_{t\to 0}\dfrac{1}{2t}\int_{x-t}^{x+t} s\,f'(s)\,ds$ for all $x\in\mathbb{R}$. Then $f(3)$ equals

(a) $\sqrt{3}$ (b) $3\sqrt{2}$ (c) $3\sqrt{3}$ (d) 9

10. Area lying in the first quadrant and bounded by the curve $y=x^3$ and $y=4x$ is

(a) 2 (b) 3 (c) 4 (d) 5

11. The area of the region bounded by $y=|x-1|$ and $y=1$ is **[IIT-JEE-1994]**

(a) 1 (b) 2 (c) 1/2 (d) None of these

12. Let $A(t)$ denote the area bounded by the curve $y=e^{-x}$, the x-axis and the straight lines $x=-t$ and $x=t$. Then $\lim\limits_{t\to\infty}A(t)$ is equal to

(a) 2 (b) 1 (c) 1/2 (d) 0

ANSWER KEY

1. 50	**2. –1**	**3. [0, 2]**	**4. ( )**	**5. (2/17)**
6. ( )	**7.** $1+\dfrac{2}{e}$	**8. (c)**	**9. (b)**	**10. (c)**
11. (a)	**12. (a)**			

EXERCISE-2

PART-A (Multiple Choice Questions (MCQ))

1. Area of the region bounded by curve $y^2(4-x)=x^3$ and its asymptote

(a) 3π (b) 12π (c) 27π (d) $\frac{3}{4}\pi$

2. The value of $\int_0^1 x^4(1-x)^3 dx$ is

(a) $\frac{1}{280}$ (b) $\frac{1}{180}$ (c) $\frac{1}{380}$ (d) $\frac{1}{80}$

3. Find the area of the segment cut off from the parabola $y^2=2x$ by the straight line $y=4x-1$ above x-axis is

(a) $\frac{7}{96}$ (b) $\frac{5}{24}$

(c) $\frac{9}{32}$ (d) None of these

4. Length of the arc of the curve $x^2=4y$ measured from the vertex to one extremity of the latus rectum

(a) $2\left[\sqrt{2}+\log\left(1+\sqrt{2}\right)\right]$

(b) $2\left[\sqrt{2}-\log\left(1+\sqrt{2}\right)\right]$

(c) $\left[\sqrt{2}+\log\left(1+\sqrt{2}\right)\right]$

(d) $\left[\sqrt{2}-\log\left(1+\sqrt{2}\right)\right]$

5. The area enclosed by the curve $y^2=x^3$ and $x^2=y^3$ is rotated about x-axis then volume of solid generated is

(a) $\frac{3\pi}{7}$ (b) $\frac{\pi}{4}$

(c) $\frac{5\pi}{28}$ (d) $\frac{19\pi}{28}$

6. Area in the positive quadrant enclosed between the four curves $y=x^3$; $4y=x^3$; $x=y^3$; $4x=y^3$ is

(a) 1 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) $\frac{1}{3}$

7. The region enclosed by the curve $y=\sin x$ and $y=\cos x$ and the x-axis between $x=0$ and $x=\frac{\pi}{2}$ is revolved about x-axis then volume of solid thus obtained

(a) $\frac{(\pi-2)}{2}$ (b) $\frac{\pi(\pi-2)}{2}$

(c) $\frac{\pi(\pi-2)}{4}$ (d) $\frac{(\pi-2)}{4}$

8. Whole area of the curve $x^2=y^3(2-y)$ is

(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{4}$ (c) $\frac{3\pi}{2}$ (d) π

9. Area of the region enclosed by $(y-1)^2=x$ and $(y-2)^2=x$ is

(a) $\frac{1}{4}$ (b) $\frac{1}{8}$ (c) $\frac{1}{12}$ (d) $\frac{1}{16}$

10. Let $f:[-1,1]\to\mathbb{R}$ be continuous differentiable function satisfying $f(x-y)=f(x)-f(y)$ $\forall\, x\in[-1,1]$ and $\int_{\sin^2 x}^{\cos^2 x} f'(2t)\,dt = x$ for $x\in\left[0,\frac{\pi}{2}\right]$

Then $\sqrt{2}f'\left(\sqrt{2}\right)$ equals

(a) 0 (b) 1 (c) $\frac{1}{2}$ (d) $-\frac{1}{2}$

11. Area bounded by the curve $y=x^4-2x^2$ and $y=2x^2$ is

(a) $\frac{112}{15}$ (b) $\frac{56}{15}$

(c) $-\frac{112}{15}$ (d) $-\frac{56}{15}$

12. Area bounded by the curve $x=3y-y^2$ and $x+y=3$ is

(a) $-\frac{4}{3}$ (b) $\frac{4}{3}$ (c) $\frac{2}{3}$ (d) $-\frac{2}{3}$

13. $f(0)=f'(0)=0$ and $f''(0)=\tan^2 x$, then $f(x)$ is

(a) $\log|\sec x|-\frac{1}{2}x^2$ (b) $\log|\cos x|+\frac{1}{2}x^2$

(c) $\log|\sec x|+\frac{1}{2}x^2$ (d) None of these

14. If $\int_{\sin x}^{1} t^2(f(t))dt = (1-\sin x)$, then $f\left(\frac{1}{\sqrt{3}}\right)$ is

(a) $\frac{1}{3}$ (b) $\frac{1}{\sqrt{3}}$ (c) 3 (d) $\sqrt{3}$

15. Let $f:[0,\infty)\to[0,\infty)$ be a twice differentiable and increasing function with $f(0)=0$, suppose that for any $l\ge 0$, the length of the arc of the curve $y=f(x), x\ge 0$ between $x=0$ and $x=l$ is $\frac{2}{3}\left[(1+l)^{3/2}-1\right]$, then $f(4)$ equal to

(a) $\frac{11}{3}$ (b) $\frac{13}{3}$ (c) $\frac{14}{3}$ (d) $\frac{16}{3}$

16. Find the area of the region bounded by curves $y=x^2$ and $y=\frac{2}{1+x^2}$

(a) $\frac{\pi}{4}-\frac{2}{3}$ (b) $\pi+\frac{2}{3}$

(c) $\pi-\frac{2}{3}$ (d) $\frac{\pi}{4}+\frac{2}{3}$

17. $\lim_{n\to\infty}\left[\frac{1}{\sqrt{2n^2+1}}+\frac{1}{\sqrt{2n^2+2}}+.....+\frac{1}{\sqrt{2n^2+n}}\right]=$

(a) 1 (b) 0 (c) $\frac{1}{\sqrt{2}}$ (d) ∞

PART-B (Multiple Select Questions (MSQ))

1. Let area of the smaller portion and larger portion enclosed by the curve $x^2+y^2=4x$; $y^2=2x$; $y\ge 0$ denoted by S and L respectively then

(a) $L-S=\frac{16}{3}$ (b) $L-S=2\pi$

(c) $L+S=\frac{16}{3}$ (d) $L+S=2\pi$

2. Let $I=\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx$ and $J=\int_0^\infty \frac{x}{1+x^6}dx$ then which of the following is true.

(a) I is convergent and $I=0$

(b) J is convergent and converges to $\frac{\pi}{3\sqrt{3}}$

(c) J is convergent and converges to $\frac{\pi}{2\sqrt{2}}$

(d) I is convergent and converges to $\sqrt{\pi}$

3. Which of the following is TRUE ?

(a) $\int_0^1 x^5(1-x^3)^{10}dx=\frac{1}{195}$

(b) $\int_0^1 x^5(1-x^3)^{10}dx=\frac{1}{196}$

(c) $\int_0^2 x(8-x^3)^{1/3}dx=\frac{32\pi}{9\sqrt{3}}$

(d) $\int_0^2 x(8-x^3)^{1/3}dx=\frac{16\pi}{9\sqrt{3}}$

4. $\int_0^x f(t)dt = x\int_x^1 f(t)dt$

(a) If $f(0)=0$, then function is even

(b) $f(-1)$ does not exist

(c) If $f(0)=1$, then $f(2)=\frac{1}{9}$

(d) If $f(0)=1$, then $f(11)=\frac{1}{44}$

5. Let $f:[a,b]\to\mathbb{R}^+$ be a continuous function bounded by k such that

L = Length of curve between a and b

A = Area bounded by curve, x-axis and $x=a$ and $x=b$

S = Area of surface generated by revolving the curve about x-axis between $x=a$ and $x=b$

Then which of the following is true ?

(a) $2\pi A<2\pi kL$ (b) $2\pi kL<S$

(c) $S\le 2\pi A$ (d) $S<2\pi kL$

6. Which of the following is TRUE ?

(a) Let f be continuously differentiable function such that $\int_0^{x^2} f(t)\,dt=e^{\sin x^2}\ \forall\ x\in(0,\infty)$ then $f(\pi)=-1$

(b) Let f be continuously differentiable function such that $\int_0^{x^2} f(t)\,dt=e^{\sin x^2}\ \forall\ x\in(0,\infty)$ then $f(\pi)=1$

(c) $f:(0,\infty)$ be differentiable function such that $f'(x^2)=1-x^3\ \forall\ x>0$ and $f(1)=K$; value of K such that $f(4)=-\frac{47}{5}$, then $K=0$.

(d) $f:(0,\infty)$ be differentiable function such that $f'(x^2)=1-x^3\ \forall\ x>0$ and $f(1)=K$; value of K such that $f(4)=-\frac{47}{5}$, then $K=1$.

7. If $f(x)=\frac{x+1}{x-1}, f^2(x)=f(f(x)), f^3(x)=f(f(f))$, $g(x)=f^{1998}(x)$, then $\int_{1/e}^{1} g(x)dx$

(a) 0 (b) 1 (c) –1 (d) e

8. $\int\sqrt{x-1}\left\{\sin^{-1}(\log x)+\cos^{-1}(\log x)\right\}dx$ if $x\in\left[\frac{1}{e},e\right]$

(a) $\frac{\pi}{3}(x-1)^{3/2}$ (b) 0

(c) does not exist (d) None of these

9. $\int\frac{1}{2\pi x-x^2}dx=fog(x)+c$

(a) $f(x)=\sin^{-1}x$ and $g(x)=\frac{x-a}{a}$

(b) $f(x)=\tan^{-1}x$ and $g(x)=\frac{x-a}{a}$

(c) fog is 1–1 $\Rightarrow g$ is 1–1; $a\in\mathbb{R}-\{0\}$

(d) fog is onto $\Rightarrow f$ is onto

PART-C (Numerical Answer Type (NAT))

1. Let $I_1=\int_0^1\frac{x^2dx}{\sqrt{1-x^4}}$; $I_2=4\sqrt{2}\int_0^1\frac{dx}{\sqrt{1-x^4}}$ then $I_1 \bullet I_2$ equals______.

2. Area of the region bounded by the curve $xy^2=4(2-x)$ and the y-axis is $4\pi K$ what is K______.

3. Area bounded by the curve $y=(x-1)^3$, x-axis and the co-ordinates $x=-1$ and $x=2$______.

4. Area bounded by the parabola $y=4x^2$ and the line $4x-y+3=0$______.

5. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function if $\int_0^x f(2t)\,dt=\frac{x}{K}\sin x$ for what value of K, $f'(2\pi)$ is -1______.

6. Let $f(x)=\int_{\cos x}^{\sin x}e^{-t^2}dt$. Then $f'(0)$ equals______.

ANSWER KEY

PART-A (Multiple Choice Questions (MCQ))

1. (b)	**2. (a)**	**3. (b)**	**4. (c)**	**5. (c)**
6. (c)	**7. (c)**	**8. (d)**	**9. (c)**	**10. (d)**
11. (a)	**12. (b)**	**13. (d)**	**14. (c)**	**15. (d)**
16. (c)	**17. (c)**			

PART-B (Multiple Select Questions (MSQ))

1. (a, d)	**2. (a, b)**	**3. (b, d)**	**4. (a, b, c)**	**5. (a, d)**
6. (a, c)	**7. (c)**	**8. (a, c)**	**9. (a, c, d)**	

PART-C (Numerical Answer Type (NAT))

1. (3.14)	**2. (1)**	**3. (4.25)**	**4. (5.5)**	**5. (1)**
6. (1)				

EXERCISE-3

PART-A (Multiple Choice Questions (MCQ))

1. The solution for x of the equation

$\int_{\sqrt{2}}^{x}\frac{dt}{t\sqrt{t^2-1}}=\frac{\pi}{2}$ is

(a) π (b) $\frac{\sqrt{3}}{2}$

(c) $2\sqrt{2}$ (d) none of these

2. Let $f(0)=0$ and $\int_0^2 f'(2t)e^{f(2t)}dt=5$. Then the value of $f(4)$ is

(a) log 2 (b) log 7

(c) log 11 (d) log 13

3. If $\int_{\log 2}^{x}\frac{dx}{\sqrt{e^x-1}}=\frac{\pi}{6}$, then x is equal to

(a) 4 (b) ln 8

(c) ln 4 (d) none of these

4. If $I_1=\int_{-100}^{101}\frac{dx}{(5+2x-2x^2)(1+e^{2-4x})}$ and

$I_2=\int_{-100}^{101}\frac{dx}{5+2x-2x^2}$, then $\frac{I_1}{I_2}$ is

(a) 2 (b) $\frac{1}{2}$ (c) 1 (d) $-\frac{1}{2}$

5. If $f(y)=e^y, g(y)=y,\ y>0,$ and

$F(t)=\int_0^t f(t-y)g(y)dt,$ then

(a) $F(t)=e^t-(1+t)$ (b) $F(t)=te^t$

(c) $F(t)=te^{-t}$ (d) $F(t)=1-e^t(1+t)$

6. The value of the definite integral

$\int_0^{\sqrt{\ln\left(\frac{\pi}{2}\right)}}\cos(e^{x^2})2xe^{x^2}dx$ is

(a) 1 (b) $1+(\sin 1)$

(c) $1-(\sin 1)$ (d) $(\sin 1)-1$

7. The value of $\int_1^{\frac{1+\sqrt{5}}{2}}\frac{x^2+1}{x^4-x^2+1}\log\left(1+x-\frac{1}{x}\right)dx$ is

(a) $\frac{\pi}{8}\log_e 2$ (b) $\frac{\pi}{2}\log_e 2$

(c) $-\frac{\pi}{2}\log_e 2$ (d) none of these

8. The numbers of possible continuous $f(x)$ defined in [0, 1] for which $I_1=\int_0^1 f(x)dx=1,$

$I_2=\int_0^1 xf(x)dx=a,\ I_3=\int_0^1 x^2 f(x)dx=a^2$ is /are

(a) 1 (b) ∞ (c) 2 (d) 0

9. $I_1=\int_0^{\frac{\pi}{2}}\ln(\sin x)dx,\ I_2=\int_{-\pi/4}^{\pi/4}\ln(\sin x+\cos x)dx.$ Then

(a) $I_1=2I_2$ (b) $I_2=2I_1$

(c) $I_1=4I_2$ (d) $I_2=4I_1$

10. If $I_1=\int_0^{\pi/2}\frac{\cos^2 x}{1+\cos^2 x}dx,\ I_2=\int_0^{\pi/2}\frac{\sin^2 x}{1+\sin^2 x}dx,$

$I_3=\int_0^{\pi/2}\frac{1+2\cos^2 x\sin^2 x}{4+2\cos^2 x\sin^2 x}dx,$ then

(a) $I_1=I_2>I_3$ (b) $I_3>I_1=I_2$ (c) $I_1=I_2=I_3$ (d) none of these

11. $I_1=\int_0^{\frac{\pi}{2}}\frac{\sin x-\cos x}{1+\sin x\cos x}dx,\ I_2=\int_0^{2\pi}\cos^6 x\,dx,$

$I_3=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin^3 x\,dx,\ I_4=\int_0^1\ln\left(\frac{1}{x}-1\right)dx.$ Then

(a) $I_2=I_3=I_4=0, I_1\neq 0$

(b) $I_1=I_2=I_3=0, I_4\neq 0$

(c) $I_1=I_3=I_4=0, I_2\neq 0$

(d) $I_1=I_2=I_3=0, I_4\neq 0$

12. If $f(x)$ and $g(x)$ are continuous functions, then $\int_{\ln\lambda}^{\ln 1/\lambda} \frac{f(x^2/4)[f(x)-f(-x)]}{g(x^2/4)[g(x)+g(-x)]}dx$ is

(a) dependent on λ
(b) a non-zero constant
(c) zero
(d) none of these

13. $\int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}dx$ is equal to

(a) π (b) π^2
(c) 0 (d) none of these

14. If $\int_{-\pi/4}^{3\pi/4} \frac{e^{\pi/4}dx}{(e^x+e^{\pi/4})(\sin x+\cos x)} = k\int_{-\pi/2}^{\pi/2} \sec x dx$, then the value of k is

(a) $\frac{1}{2}$ (b) $\frac{1}{\sqrt{2}}$ (c) $\frac{1}{2\sqrt{2}}$ (d) $-\frac{1}{\sqrt{2}}$

15. $\int_{-\pi/3}^{0}\left[\cot^{-1}\left(\frac{2}{2\cos x-1}\right)+\cot^{-1}\left(\cos x-\frac{1}{2}\right)\right]dx$ is equal to

(a) $\frac{\pi^2}{6}$ (b) $\frac{\pi^2}{3}$ (c) $\frac{\pi^2}{8}$ (d) $\frac{3\pi^2}{8}$

16. $\int_{0}^{\infty}\left(\frac{\pi}{1+\pi^2x^2}-\frac{1}{1+x^2}\right)\log x\,dx$ is equal to

(a) $-\frac{\pi}{2}\ln\pi$ (b) 0
(c) $\frac{\pi}{2}\ln 2$ (d) none of these

17. The equation of the curve is $y=f(x)$. The tangents at $[1, f(1)], [2, f(2)]$, and $[3, f(3)]$ make angles $\frac{\pi}{6}, \frac{\pi}{3}$, and $\frac{\pi}{4}$, respectively, with the positive direction of x-axis. Then the value of $\int_2^3 f'(x)f''(x)dx + \int_1^3 f''(x)dx$ is equal to

(a) $-1/\sqrt{3}$ (b) $1/\sqrt{3}$
(c) 0 (d) none of these

18. If $f(\pi)=2$ and $\int_0^{\pi}(f(x)+f''(x))\sin x\,dx=5$, then $f(0)$ is equal to (it is given that $f(x)$ is continuous in $[0, \pi]$)

(a) 7 (b) 3 (c) 5 (d) 1

19. If $\int_1^2 e^{x^2}dx = a$, then $\int_e^{e^4}\sqrt{\ln x}\,dx$ is equal to

(a) $2e^4-2e-a$ (b) $2e^4-e-a$
(c) $2e^4-e-2a$ (d) e^4-e-a

20. $\int_{-\pi/2}^{\pi/2}\frac{e^{|\sin x|}\cos x}{(1+e^{\tan x})}dx$ is equal to

(a) $e+1$ (b) $1-e$
(c) $e-1$ (d) none of these

21. If $A=\int_0^{\pi}\frac{\cos x}{(x+2)^2}dx$, then $\int_0^{\pi/2}\frac{\sin 2x}{x+1}dx$ is equal to

(a) $\frac{1}{2}+\frac{1}{\pi+2}-A$ (b) $\frac{1}{\pi+2}-A$
(c) $1+\frac{1}{\pi+2}-A$ (d) $A-\frac{1}{2}-\frac{1}{\pi+2}$

22. Let $I_1=\int_0^1\frac{e^x dx}{1+x}$ and $I_2=\int_0^1\frac{x^2dx}{e^{x^3}(2-x^3)}$. Then $\frac{I_1}{I_2}$ is equal to

(a) $3/e$ (b) $e/3$ (c) $3e$ (d) $1/3e$

23. If $\int_0^1\frac{\sin t}{1+t}dt=\alpha$, then the value of the integral $\int_{4\pi-2}^{4\pi}\frac{\sin\frac{t}{2}}{4\pi+2-t}dt$ is

(a) 2α (b) -2α (c) α (d) $-\alpha$

24. If $I(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx, (m,n\in I, m,n\ge 0)$, then

(a) $I(m,n)=\int_0^{\infty}\frac{x^{m-1}}{(1+x)^{m-n}}dx$

(b) $I(m,n)=\int_0^{\infty}\frac{x^{m}}{(1+x)^{m+n}}dx$

(c) $I(m,n)=\int_0^{\infty}\frac{x^{n-1}}{(1+x)^{m+n}}dx$

(d) $I(m,n)=\int_0^{\infty}\frac{x^{n}}{(1+x)^{m+n}}dx$

25. The value of $\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} dx$ is, $n \in I, n \geq 0$,

(a) $\frac{\pi}{2}$ (b) 0 (c) π (d) 2π

26. The value of the definite integral $\int_0^{\pi/2} \frac{\sin 5x}{\sin x} dx$ is

(a) 0 (b) $\frac{\pi}{2}$ (c) π (d) 2π

27. If $I_n = \int_0^{\pi} e^x (\sin x)^n dx$, then $\frac{I_3}{I_1}$ is equal to

(a) 3/5 (b) 1/5 (c) 1 (d) 2/5

28. The value of $\int_0^{\pi/2} \sin|2x-\alpha|\, dx$, where $\alpha \in [0,\pi]$, is

(a) $1-\cos\alpha$ (b) $1+\cos\alpha$
(c) 1 (d) $\cos\alpha$

29. Let $f(x) = \min(\{x\}, \{-x\}) \forall x \in R$, where $\{\cdot\}$ denotes the fractional part of x. Then $\int_{-100}^{100} f(x)dx$ is equal to

(a) 50 (b) 100
(c) 200 (d) none of these

30. $\int_1^4 \{x-0.4\}dx$ equals (where $\{x\}$ is a fractional part of x)

(a) 13 (b) 6.3 (c) 1.5 (d) 7.5

31. The value of

$\int_0^x [\cot t]dt, x \in \left[(4n+1)\frac{\pi}{2}, (4n+3)\frac{\pi}{2}\right]$ and $n \in N$, is equal to (where $[\cdot]$ represents greatest integer function)

(a) $\frac{\pi}{2}(2n-1)-2x$ (b) $\frac{\pi}{2}(2n-1)+x$

(c) $\frac{\pi}{2}(2n+1)-x$ (d) $\frac{\pi}{2}(2n+1)+x$

32. If $f(x) = \int_0^1 \frac{dt}{1+|x-t|}$, then $f'\left(\frac{1}{2}\right)$ is equal to

(a) 0 (b) $\frac{1}{2}$
(c) 1 (d) none of these

33. The value of the definite integral

$$\int_2^4 (x(3-x)(4+x)(6-x)(10-x)+\sin x)dx$$

equals

(a) $\cos 2+\cos 4$ (b) $\cos 2-\cos 4$
(c) $\sin 2+\sin 4$ (d) $\sin 2-\sin 4$

34. If $f(x) = \cos x - \int_0^x (x-t)f(t)dt$, then $f'(x)+f(x)$ is equal to

(a) $-\cos x$ (b) $-\sin x$

(c) $\int_0^x (x-t)f(t)dt$ (d) 0

35. If $\int_0^x f(t)dt = x + \int_x^1 tf(t)dt$, then the value of $f(1)$ is

(a) 1/2 (b) 0
(c) 1 (d) –1/2

36. If $\int_{\cos x}^1 t^2 f(t)dt = 1-\cos x \; \forall x \in \left(0,\frac{\pi}{2}\right)$, then the value of $\left[f\left(\frac{\sqrt{3}}{4}\right)\right]$ is ($[\cdot]$ denotes the greatest integer function)

(a) 4 (b) 5 (c) 6 (d) –7

37. If $\int_0^{f(x)} t^2 dt = x\cos\pi x$, then $f'(9)$ is

(a) $-\frac{1}{9}$ (b) $-\frac{1}{3}$

(c) $\frac{1}{3}$ (d) non-existent

38. If $f(x) = 1+\frac{1}{x}\int_1^x f(t)dt$, then the value of $f(e^{-1})$ is

(a) 1 (b) 0
(c) –1 (d) none of these

39. $\int_0^x [\sin t]\,dt$, where $x \in (2n\pi, (2n+1)\pi), n \in N$, and $[\cdot]$ denotes the greatest integer function, is equal to

(a) $-n\pi$ (b) $-(n+1)\pi$

(c) $-2n\pi$ (d) $-(2n+1)\pi$

40. $f(x)$ is a continuous function for all real values of x and satisfies $\int_0^x f(t)dt = \int_x^1 t^2 f(t)dt + \frac{x^{16}}{8} + \frac{x^6}{3} + a$. Then the value of a is equal to

(a) $-\frac{1}{24}$ (b) $\frac{17}{168}$

(c) $\frac{1}{7}$ (d) $-\frac{167}{840}$

41. $f(x)$ is a continuous function for all real values of x and satisfies $\int_n^{n+1} f(x)\,dx = \frac{n^2}{2}\ \forall n \in I$. Then $\int_{-3}^{5} f(|x|)\,dx$ is equal to

(a) 19/2 (b) 35/2

(c) 17/2 (d) none of these

42. $\int_{-3}^{3} x^8\{x^{11}\}dx$ is equal to (where $\{\cdot\}$ is the fractional part of x)

(a) 3^8 (b) 3^7

(c) 3^9 (d) none of these

43. f is an odd function. It is also known that $f(x)$ is continuous for all values of x and is periodic with periodic with periodic 2. If $g(x) = \int_0^x f(t)dt$, then

(a) $g(x)$ is odd

(b) $g(n) = 0,\ n \in N$

(c) $g(2n) = 0,\ n \in N$

(d) $g(x)$ is non-periodic

44. $\int_0^x |\sin t|\,dt$, where $x \in (2n\pi, (2n+1)\pi), n \in N$, is equal to

(a) $4n - \cos x$ (b) $4n - \sin x$ (c) $4n + 1 - \cos x$ (d) $4n - 1 - \cos x$

45. If $a > 0$ and $A = \int_0^a \cos^{-1} x\,dx$, then $\int_{-a}^{a} (\cos^{-1} x - \sin^{-1}\sqrt{1-x^2})dx = \pi a - \lambda A$. Then λ is

(a) 0 (b) 2

(c) 3 (d) none of these

46. Let f be integrable over $[0, a]$ for any real value of a. If $I_1 = \int_0^{\pi/2} \cos\theta f(\sin\theta + \cos^2\theta)d\theta$ and $I_2 = \int_0^{\pi/2} \sin 2\theta f(\sin\theta + \cos^2\theta)d\theta$, then

(a) $I_1 = -2I_2$ (b) $I_1 = I_2$

(c) $2I_1 = I_2$ (d) $I_1 = -I_2$

47. If $f'(x) = f(x) + \int_0^1 f(x)\,dx$, given $f(0) = 1$, then the value of $f(\log_e 2)$ is

(a) $\frac{1}{3+e}$ (b) $\frac{5-e}{3-e}$

(c) $\frac{2+e}{e-2}$ (d) none of these

48. The functions f and g are positive and continuous. If f is increasing and g is decreasing, then $\int_0^1 f(x)[g(x) - g(1-x)]dx$

(a) is always non-positive

(b) is always non-negative

(c) can take positive and negative values

(d) none of these

49. If $f(x) = \begin{Bmatrix} 0, \text{where } x = \frac{n}{n+1}, n = 1, 2, 3, ... \\ 1, \text{elsewhere} \end{Bmatrix}$, then the value of $\int_0^2 f(x)dx$ is

(a) 1 (b) 0 (c) 2 (d) ∞

50. Let $f(x)$ be positive, continuous, and differentiable on the interval (a, b) an $\lim_{x\to a^+} f(x) = 1$, $\lim_{x\to b^-} f(x) = 3^{1/4}$. If $f'(x) \geq f^3(x) + \frac{1}{f(x)}$, then the greatest value of b-a is

(a) $\frac{\pi}{48}$ (b) $\frac{\pi}{36}$ (c) $\frac{\pi}{24}$ (d) $\frac{\pi}{12}$

51. Consider the Region R defined as

$$R=\left\{(x,y)\left|\begin{matrix}x^2+y^2\le 100\\ \sin(x+y)\ge 0\end{matrix}\right.\right\}$$

What is the area of R?

(a) 25π (b) 40π (c) 50π (d) 64π

52. A curve $y=f(x)$ passes through the origin through any point (x,y) on the curve, lines are drawn parallel to the co-ordinate axes. If the curve bounded, the area formed by these lines and the co-ordinate axes in the ratio m/n. What is the equation of curve? In the following options, k represents an arbitrary constant.

(a) $y^2=kx^{m/n}$ (b) $y=kx^{m/n}$

(c) $y=kx^{2m/n}$ (d) $y=kx^{\frac{1+m}{n}}$

53. What is the positive value of b for which the area of the bounded region bounded enclosed between the parabolas $y=x-bx^2$ and $y=\dfrac{x^2}{b}$ is maximum?

(a) 1 (b) $\sqrt{2}$ (c) $\sqrt{3}$ (d) 2

54. Let $f(x)=Max\{x^3,(1-x)^2,2x(1-x)\}$ what is the area of the region bounded by the curve $y=f(x)$, $x=0$ and $x=1$?

(a) $\dfrac{58}{27}$ (b) $\dfrac{61}{27}$ (c) $\dfrac{67}{27}$ (d) $\dfrac{71}{27}$

55. What is the area of the region bounded between the curves $y=e^x\ln x$ and $y=\dfrac{\ln x}{ex}$

(a) $\dfrac{e^2-1}{2e}$ (b) $\dfrac{e^2-3}{2e}$

(c) $\dfrac{e^2-5}{4e}$ (d) $\dfrac{e^2-6}{5e}$

56. What is the area of the region containing the points where (x,y) co-ordinates satisfy the following relation $|y|+\dfrac{1}{2}\le e^{-|x|}$

(a) $1-\ln 2$ (b) $1+\ln 2$

(c) $2(1-\ln 2)$ (d) $2(1+\ln 2)$

57. A square has its vertices at (1, 1) (1, –1), (–1, –1) and (–1, 1). Four circles of radius 2 are drawn one centred at each vertex of the square. What is the area common to these four circles.

(a) $2\left(\dfrac{\pi}{3}-\sqrt{3}\right)$ (b) $4\left(\dfrac{\pi}{3}-\sqrt{3}\right)$

(c) $2\left(\dfrac{\pi}{3}+\dfrac{1}{\sqrt{3}}\right)$ (d) $4\left(\dfrac{\pi}{3}+\dfrac{1}{\sqrt{3}}\right)$

58. What is the area of the region bounded by the curves $y=x^2$ $y=|2-x^2|$ and $y=2$ which lies to the right of the line $x=1$?

(a) $\dfrac{10-4\sqrt{2}}{3}$ (b) $\dfrac{12-5\sqrt{2}}{3}$

(c) $\dfrac{16-8\sqrt{2}}{3}$ (d) $\dfrac{20-12\sqrt{2}}{3}$

59. What is the value of $I=\int_0^\infty e^{-ax}x^n dx$

(a) $\dfrac{(n-1)!}{a^n}$ (b) $\dfrac{n!}{a^n}$

(c) $\dfrac{(n-1)!}{a^{n+1}}$ (d) $\dfrac{n!}{a^{n+1}}$

60. What is the value $I=\int_0^1 \dfrac{x^\alpha-x^\beta}{\ln x}dx$

(a) $\ln\left(\dfrac{\alpha}{\beta}\right)$ (b) $2\ln\left(\dfrac{\alpha}{\beta}\right)$

(c) $\ln\left(\dfrac{\alpha+1}{\beta+1}\right)$ (d) $\ln\left(\dfrac{\alpha-1}{\beta-1}\right)$

61. What is the value of $I=\int_0^\infty \dfrac{\tan^{-1}a^x}{x(1+x^2)}dx\, a\ge 0$?

(a) $\dfrac{\pi}{4}\ln(1+a)$ (b) $\dfrac{\pi}{3}\ln(1+a)$

(c) $\dfrac{\pi}{2}\ln(1+a)$ (d) $\pi\ln(1+a)$

62. Area bounded by the curves $y = \tan x$ and $y = \tan^2 x$ in between $x \in \left(-\frac{\pi}{3}, \frac{\pi}{3}\right)$ is equal to

(a) $\frac{1}{2}(\pi + \ln 2 - 2)$ sq. units

(b) $\frac{1}{3}\{\pi + \ln(2\sqrt{2} - 3)\}$ sq. units

(c) $\left[\frac{\pi}{6} + \ln 2 + 2\sqrt{3} - 2\right]$ sq. units

(d) $\frac{1}{2}(\pi + \ln 4 - 2)$ sq. units

63. Area enclosed by the curve $(y - \sin^{-1} x)^2 = x - x^2$ is equal to

(a) $\frac{\pi}{2}$ sq. units (b) $\frac{\pi}{4}$ sq. units

(c) $\frac{\pi}{8}$ sq. units (d) None of these

64. The area of the loop of the curve $x^2 + (y-1)y^2 = 0$ equal to

(a) $\frac{8}{15}$ sq. units (b) $\frac{8}{17}$ sq. units

(c) $\frac{4}{15}$ sq. units (d) None of these

65. Area of the region which consists of all the points satisfying the deduction $|x - y| + |x + y| \le 8$ and $xy \ge 2$ is equal to

(a) $4(7 - \ln 8)$ sq. units

(b) $4(9 - \ln 8)$ sq. units

(c) $2(7 - \ln 8)$ sq. units

(d) $2(9 - \ln 8)$ sq. units

66. A point P moves in xy plane in such away that $[x + y + 1] = [x]$ where $[x]$ represents the greatest integer formation $x \in (0, 2)$. Area of the region representing all possible positions of the point 'P' is equal to

(a) 2 sq. units (b) 8 sq. units

(c) $\sqrt{2}$ sq. units (d) 4 sq. units

67. A point $P(x, y)$ moves in xy plane in such away take $\sqrt{2} \le |x + y| + |x - y| \le 2\sqrt{2}$ area of the region representing all possible positions of the plant 'P' is equal to

(a) 2 sq. units (b) 4 sq. units

(c) 6 sq. units (d) 8 sq. units

68. Two lines drawn through the point P(4, 0) divide. The area bounded by the curve $y = \sqrt{2} \sin \frac{\pi x}{4}$ and x-axis between the lines $x = 2$ and $x = 4$ into three equal parts sum of the slopes of the down lines is equal to

(a) $\frac{-2\sqrt{2}}{\pi}$ (b) $\frac{-\sqrt{2}}{\pi}$

(c) $\frac{-2}{\pi}$ (d) $\frac{-4\sqrt{2}}{\pi}$

69. The area of the smaller region in which the curve $y = \left[\frac{x^3}{100} + \frac{x}{50}\right]$ where $[\cdot]$ denotes the greatest integer function, divide the circle $(x-2)^2 + (y+1)^2 = 4$ is equal to

(a) $\frac{2\pi - 3\sqrt{3}}{3}$ sq. units

(b) $\frac{3\sqrt{3} - \pi}{3}$ sq. units

(c) $\frac{4\pi - 3\sqrt{3}}{3}$ sq. units

(d) $\frac{5\pi - 3\sqrt{3}}{3}$ sq. units

70. The area of the figure bounded by $y^2 = (2x + 1)$ and $(x - y - 1) = 0$

(a) 2/3 (b) 4/3

(c) 8/3 (d) 16/3

PART-B (Multiple Select Questions (MSQ))

1. A function $f(x)$ satisfies the relation $f(x) = e^x + \int_0^1 e^x f(t)dt$. Then

(a) $f(0) < 0$

(b) $f(x)$ is a decreasing function

(c) $f(x)$ is an increasing function

(d) $\int_0^1 f(x)dx > 0$

2. Let $f(x) = \int_1^x \frac{3^t}{1+t^2} dt$, where $x > 0$. Then

(a) for $0 < \alpha < \beta, f(\alpha) < f(\beta)$

(b) for $0 < \alpha < \beta, f(\alpha) > f(\beta)$

(c) $f(x) + \pi/4 < \tan^{-1} x \,\forall\, x \geq 1$

(d) $f(x) + \pi/4 > \tan^{-1} x \,\forall\, x \geq 1$

3. The values of a for which the integral $\int_0^2 |x-a|\,dx \geq 1$ is satisfied are

(a) $[2, \infty)$ (b) $(-\infty, 0]$ (c) (0, 2) (d) none of these

4. If $\int_a^b |\sin x|\,dx = 8$ and $\int_0^{a+b} |\cos x|\,dx = 9$, then

(a) $a + b = \frac{9\pi}{2}$ (b) $|a - b| = 4\pi$

(c) $\frac{a}{b} = 15$ (d) $\int_a^b \sec^2 x dx = 0$

5. Let $I = \int_1^3 \sqrt{3+x^3}\,dx$, then the values of I will lie in the interval

(a) [4, 6] (b) [1, 3]

(c) $[4, 2\sqrt{30}]$ (d) $[\sqrt{15}, \sqrt{30}]$

6. If $g(x) = \int_0^x 2|t|\,dt$, then

(a) $g(x) = x|x|$

(b) $g(x)$ is monotonic

(c) $g(x)$ is differentiable at $x = 0$

(d) $g'(x)$ is differentiable at $x = 0$

7. Let $f:[1, \infty) \to R$ and $f(x) = x\int_1^x \frac{e^t}{t} dt - e^x$. Then

(a) $f(x)$ is an increasing function

(b) $\lim_{x\to\infty} f(x) \to \infty$

(c) $f'(x)$ has a maxima at $x = e$

(d) $f(x)$ is a decreasing function

8. The value of $\int_0^\infty \frac{dx}{1+x^4}$ is

(a) same as that of $\int_0^\infty \frac{x^2+1dx}{1+x^4}$

(b) $\frac{\pi}{2\sqrt{2}}$

(c) same as that of $\int_0^\infty \frac{x^2 dx}{1+x^4}$

(d) $\frac{\pi}{\sqrt{2}}$

9. If $f(x) = \int_0^x |t-1|\,dt$, where $0 \leq x \leq 2$, then

(a) range of $f(x)$ is [0, 1]

(b) $f(x)$ is differentiable at $x = 1$

(c) $f(x) = \cos^{-1} x$ has two real roots

(d) $f'(1/2) = 1/2$

10. If $I_n = \int_0^{\pi/4} \tan^n x\,dx$, ($n > 1$ and is an integer), then

(a) $I_n + I_{n-2} = \frac{1}{n+1}$

(b) $I_n + I_{n-2} = \frac{1}{n-1}$

(c) $I_2 + I_4, I_4 + I_6, \ldots,$ are in H.P.

(d) $\frac{1}{2(n+1)} < I_n < \frac{1}{2(n-1)}$

11. If $I_n = \int_0^1 \frac{dx}{(1+x^2)^n}$, where $n \in N$, which of the following statements hold good?

(a) $2n I_{n+1} = 2^{-n} + (2n-1)I_n$

(b) $I_2 = \frac{\pi}{8} + \frac{1}{4}$

(c) $I_2 = \frac{\pi}{8} - \frac{1}{4}$

(d) $I_3 = \frac{3\pi}{32} + \frac{1}{4}$

12. If $f(2-x)=f(2+x)$ and $f(4-x)=f(4+x)$ for all x and $f(x)$ is a function for which $\int_0^2 f(x)dx=5,$ then $\int_0^{50} f(x)dx$ is equal to

(a) 125 (b) $\int_{-4}^{46} f(x)dx$

(c) $\int_1^{51} f(x)dx$ (d) $\int_2^{52} f(x)dx$

13. $\int_0^x\left\{\int_0^u f(t)dt\right\}du$ is equal to

(a) $\int_0^x (x-u)f(u)du$

(b) $\int_0^x uf(x-u)du$

(c) $x\int_0^x f(u)du$

(d) $x\int_0^x uf(u-x)du$

14. Which of the following statement(s) is/are true?

(a) If function $y=f(x)$ is continuous at $x=c$ such that $f(c)\neq 0,$ then $f(x)f(c)>0\,\forall\, x\in(c-h,c+h),$ where h is sufficiently small positive quantity

(b) $\lim\limits_{n\to\infty}\frac{1}{n}\ln\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)...\left(1+\frac{n}{n}\right)\right)=1+2\ln 2$

(c) Let f be a continuous and non-negative function defined on $[a,b]$. If $\int_a^b f(x)dx=0,$ then $f(x)=0\,\forall\, x\in[a,b]$

(d) Let f be a continuous function defined on $[a,b]$ such that $\int_a^b f(x)dx=0$. Then there exists at least one $c\in(a,b)$ for which $f(c)=0$

15. The value of $\int_0^1 e^{x^2-x}dx$ is

(a) <1 (b) >1 (c) $>e^{-\frac{1}{4}}$ (d) $<e^{-\frac{1}{4}}$

16. For which of the following values of m, is the area of the regions bounded by the curve $y=x-x^2$ and the line $y=mx$ equals $\frac{9}{2}$

(a) –4 (b) –2 (c) 2 (d) 4

17. Area of the region bounded by the curves $y=e^x$ and lines $x=0$ and $y=e$ is

(a) $(e-1)$ (b) $\int_1^e \ln(e+1-y)dy$

(c) $e-\int_0^1 e^x dx$ (d) $\int_1^e \ln y\, dy$

18. If S be the area of the region enclosed $y=e^{-x^2}, y=0, x=0$ and $x=1$ then

(a) $S\geq\frac{1}{e}$ (b) $S\geq 1-\frac{1}{e}$

(c) $S\leq\frac{1}{4}\left(1+\frac{1}{\sqrt{e}}\right)$

(d) $S\leq\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{e}}\left(1-\frac{1}{\sqrt{e}}\right)$

19. Let $A(k)$ be the area bounded by the curves $y=(x^2-3)$ and $y=(kx+2)$

(a) the range of $A(k)$ is $\left[\frac{10\sqrt{5}}{3},\infty\right]$

(b) the range of $A(k)$ is $\left[\frac{20\sqrt{5}}{3},\infty\right]$

(c) if function $K\to A(k)$ is defined for $k\in[-2,\infty]$ then $A(k)$ is many one function

(d) the value of k for which area is minimum is 1

20. If the curve $y=ax^{1/2}+bx$ passes through the point (1, 2) and lies above the x-axis for $0\leq x\leq 9$ and the area enclosed by the curve, the x-axis, and the line $x=4$ is 8 sq. units, then

(a) $a=1$ (b) $b=1$

(c) $a=3$ (d) $b=-1$

PART-C (Numerical Answer Type (NAT))

1. Consider the polynomial $f(x) = ax^2 + bx + c$. If $f(0) = 0, f(2) = 2,$ then the minimum value of $\int_0^2 |f'(x)|\,dx$ is _________

2. Consider a real-valued continuous function *f* such that $f(x) = \sin x + \int_{-\pi/2}^{\pi/2} (\sin x + tf(t))dt$. If *M* and *m* are maximum and minimum values of the function *f*, then the value of *M*/*m* is________

3. A continuous real function *f* satisfies $f(2x) = 3f(x) \forall x \in R$. If $\int_0^1 f(x)dx = 1,$ then the value of definite integral $\int_1^2 f(x)dx$ is______

4. Let $f:[0,\infty) \to R$ be a continuous strictly increasing function, such that $f^3(x) = \int_0^x t \cdot f^2(t)\,dt$ for every $x \ge 0$. Then value of *f*(6) is_______

5. If $I = \int_0^{3\pi/5} ((1+x)\sin x + (1-x)\cos x)dx,$ then the value of $(\sqrt{2}-1)I$ is__________

6. Let $f(x) = \int_0^x \frac{dt}{\sqrt{1+t^3}}$ and $g(x)$ be the inverse of $f(x)$. Then the value of $4\frac{g''(x)}{(g(x))^2}$ is_______

7. If $\int_0^\infty x^{2n+1} \cdot e^{-x^2 dx} = 360,$ then the value of *n* is____

8. Let $f(x)$ be a derivable function satisfying $f(x) = \int_0^x e^t \sin(x-t)dt$ and $g(x) = f''(x) - f(x)$. Then the possible integers in the range of $g(x)$ is_________

9. If $F(x) = \frac{1}{x^2}\int_4^x [4t^2 - 2F'(t)]dt,$ then $(9F'(4))/4$ is________

10. If $f(x) = x + \int_0^1 t(x+t)f(t)dt,$ then the value of $\frac{23}{2}f(0)$ is equal to_______

11. The area enclosed by the curve $c: y = x\sqrt{9-x^2}\ (x \ge 0)$ and the *x*-axis is_____

12. If the area enclosed by the curve $y = \sqrt{x}$ and $x = -\sqrt{y}$ the circle $x^2 + y^2 = 2$ above the *x*-axis is A, then the value $\frac{16}{\pi}A$ is________

13. If the area bounded by the curve $y = (x^2+1)$ and the tangents to it drawn from the origin is A, then the value of 3A is _______

14. If A is the area bounded by the curves $y = \sqrt{1-x^2}$ and $y = x^3 - x$, then the value of $\frac{\pi}{A}$ ________

15. The area of the region $\{(x,y): 0 \le y \le x^2+1, 6 \le y \le x+1, 0 \le x \le 2\}$ is A then the value of $3A-17$ is _______

ANSWER KEY

PART-A (Multiple Choice Questions (MCQ))

1. (d)	**2.** (c)	**3.** (c)	**4.** (b)	**5.** (a)
6. (c)	**7.** (a)	**8.** (d)	**9.** (a)	**10.** (c)
11. (c)	**12.** (c)	**13.** (b)	**14.** (c)	**15.** (a)
16. (a)	**17.** (a)	**18.** (b)	**19.** (b)	**20.** (c)
21. (a)	**22.** (c)	**23.** (d)	**24.** (c)	**25.** (c)
26. (b)	**27.** (a)	**28.** (c)	**29.** (a)	**30.** (c)
31. (c)	**32.** (a)	**33.** (b)	**34.** (a)	**35.** (a)
36. (b)	**37.** (a)	**38.** (b)	**39.** (a)	**40.** (d)
41. (b)	**42.** (b)	**43.** (c)	**44.** (c)	**45.** (b)
46. (b)	**47.** (b)	**48.** (a)	**49.** (c)	**50.** (c)
51. (c)	**52.** (b)	**53.** (a)	**54.** (d)	**55.** (c)
56. (c)	**57.** (b)	**58.** (d)	**59.** (d)	**60.** (c)
61. (d)	**62.** (c)	**63.** (b)	**64.** (a)	**65.** (a)
66. (a)	**67.** (c)	**68.** (a)	**69.** (c)	**70.** (d)

PART-B (Multiple Select Questions (MSQ))

1. (a, b)	**2.** (a, d)	**3.** (a, b, c)	**4.** (a, b)	**5.** (c)
6. (a, b, c)	**7.** (a, b)	**8.** (b, c)	**9.** (a, b, d)	**10.** (b, c, d)
11. (a, b, d)	**12.** (a, b, d)	**13.** (a, b)	**14.** (a, c, d)	**15.** (a, d)
16. (b, d)	**17.** (b, c, d)	**18.** (a, b, d)	**19.** (b, c)	**20.** (c, d)

PART-C (Numerical Answer Type (NAT))

1. (2)	**2.** (3)	**3.** (5)	**4.** (6)	**5.** (2)
6. (6)	**7.** (6)	**8.** (3)	**9.** (8)	**10.** (9)
11. (9)	**12.** (8)	**13.** (2)	**14.** (2)	**15.** (6)