Chapter 6

NORMAL SUBGROUP

Normal Subgroup:

Let G be a group under multiplication and H be any subgroup of G and let $x \in G$. Then Hx and xH are respectively the right and left cosets of H in G.

If G is abelian then $Hx = xH \ \forall x \in G$.

But even when G is non abelian and yet there exist a subgroup H of G having the property $Hx = xH \ \forall x \in G$, then such a subgroup of G is called normal subgroup.

A normal subgroup H of a group G is denoted by $H \triangleright G$

Definition: A subgroup H of G is called normal subgroup of G if $xhx^{-1} \in H$ for all $x \in G$ and for all $h \in H$.

Note: For a group $G, \{e\}$ and G are always the normal subgroups of G and these are called trivial normal subgroups of G or improper normal subgroups of G.

Simple Group: If a group G has no proper normal subgroup, then G is called a simple group

Note: Every group of prime order is simple.

Theorem 1: Every subgroup of an abelian group is always normal

Theorem 2: A subgroup H of a group G is normal if and only if $xHx^{-1} = H$ for all $x \in G$

Theorem 3: A subgroup H of a group G is normal if and only if each left coset of H in G is a right coset of H in G.

Theorem 4: A subgroup H of G is normal if and only if the product of two right cosets of H in G is again a right cosets of H in G.

Theorem5: Intersection of two normal subgroups of a group is also a normal subgroup

Theorem6: Intersection of any collection of normal subgroups is itself a normal subgroup

Theorem7: If M and N are two normal subgroups of G such that, $N \cap M = \{e\}$, then for every $n \in N$ and $m \in N$ we have $nm = mn$.

Theorem 8: Let H be a subgroup of G and N be a normal subgroup of G. Then $H \cap N$ is a subgroup of H.

Solved Examples

1. If H is a subgroup of G and N is a normal subgroup of G, then $H \cap N$ is a normal subgroup of

(a) H (b) N (c) $H+N$ (d) G **[B.H.U.-2011]**

Soln. Let $x \in H \cap N$

$\Rightarrow x \in H$ and $x \in N$

Let $h \in H \Rightarrow h \in G$

$\Rightarrow hxh^{-1} \in H$ ($\because$ H is subgroup of G)

Also $hxh^{-1} \in N$ ($\because$ N is normal subgroup of G)

$\Rightarrow hxh^{-1} \in H \cap N \;\; \forall h \in H, \forall x \in H \cap N$

$H \cap N$ is normal subgroup of H.

Hence correct option is (a)

2. Let H be a finite subgroup of a group G and let $g \in G$. If $gHg^{-1} = \{ghg^{-1} \mid h \in H\}$, then **[B.H.U-2018]**

(a) $|gHg^{-1}| = |H|$ (b) $|gHg^{-1}| < |H|$

(c) $|gHg^{-1}| > |H|$ (d) $|gHg^{-1}| = 1$

Soln. Define $f : H \to gHg^{-1}$ by $f(h) = ghg^{-1}, h \in H$

Let $h_1, h_2 \in H$

$f(h_1h_2) = gh_1h_2g^{-1} = gh_1g^{-1}gh_2g^{-1} = f(h_1)f(h_2)$

$\Rightarrow f$ is a homomorphism

Let $f(h_1) = f(h_2)$

$\Rightarrow gh_1g^{-1} = gh_2g^{-1}$

$\Rightarrow h_1 = h_2$

$\Rightarrow f$ is one- one

Let $x \in gHg^{-1}$

$\Rightarrow x = gh_1g^{-1}$ for some $h_1 \in H$

Now $h_1 \in H$

$\Rightarrow f(h_1) = gh_1g^{-1}$

$\Rightarrow f$ is onto

$\Rightarrow f$ is one-one and onto

$\Rightarrow |gHg^{-1}| = |H|$

Hence correct option is (a)

3. Let G be a finite group and H is a subgroup of G of index 2. Then **[H.C.U. 2018]**

(a) H is normal and $g^2 \in H$ for any $g \in G$ (b) H is normal and $g^2 = e$

(c) H is need not be normal (d) None of the above

Soln. Given H is a subgroup of G of index 2

$\Rightarrow H$ is normal in G and also $(gH)^2 = H \;\; \forall g \in G$ (it is normal because every right coset is also a left coset)

$\Rightarrow (gH)(gH) = H$

$\Rightarrow g^2 H = H$ ($\because$ H is normal in G)

$\Rightarrow g^2 \in H$ ($\because$ $h \in H \Leftrightarrow Hh = H = hH$)

Hence correct option is (a)

4. Let $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{Q} - \{0\}, b \in \mathbb{Q} \right\}, U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{Q} \right\}, D = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{Q} - \{0\} \right\}$

Which of the following statements are true? **[H.C.U. 2013]**

(a) G, U, D are all groups under multiplication (b) D is a normal subgroup of G

(c) U is a normal subgroup of G (d) For every matrix $A \in U$, $ADA^{-1} \subseteq D$

Soln. Given $G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{Q} - \{0\}, b \in \mathbb{Q} \right\}$

$U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| b \in \mathbb{Q} \right\}, D = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \middle| a \in \mathbb{Q} - \{0\} \right\}$

We can easily prove that G, U,D are all groups under multiplication.

Also $U \subseteq G$ and $D \subseteq G$

$\Rightarrow U$ and D are subgroups of G.

To check that D is normal in G or not :

Let $B \in G$ and $C \in D$

$\Rightarrow B = \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix}$ for some $a_1 \in \mathbb{Q} - \{0\}, b_1 \in \mathbb{Q}$ and $C = \begin{pmatrix} a_3 & 0 \\ 0 & 1 \end{pmatrix}$ for some $a_3 \in \mathbb{Q} - \{0\}$

Now $BCB^{-1} = \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a_3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{a_1} & \frac{-b_1}{a_1} \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{a_3}{a_1} & \frac{-a_3 b_1}{a_1} \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} a_3 & -a_3 b_1 + b_1 \\ 0 & 1 \end{pmatrix} \notin D$$

$\Rightarrow D$ is not normal in G.

To check that U is normal in G or not

Let $B \in G$ and $C \in U$

$$\Rightarrow B = \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} \text{ for some } a_1 \in \mathbb{Q} - \{0\}, b_1 \in \mathbb{Q} \text{ and } C = \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}; b_2 \in \mathbb{Q}$$

$$BCB^{-1} = \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{a_1} & -\frac{b_1}{a_1} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{a_1} & \frac{-b_1}{a_1} + b_2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -b_1 + a_1 b_2 + b_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a_1 b_2 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow BCB^{-1} \in U$

$\Rightarrow U$ is normal in G.

Let $A \in U$

$$\Rightarrow A = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}; b \in \mathbb{Q}$$

Now $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & -ab \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & -ab + b \\ 0 & 1 \end{pmatrix} \notin D$

$\Rightarrow ADA^{-1} \not\subseteq D$

Correct option is (a) and (c)

5. For a group G, which of the following statements are true? **[H.C.U. 2014]**

(a) If $x, y \in G$ such that order of x is 3, order of y is 2 then order of xy is 6.

(b) If every element is of finite order in G then G is a finite group

(c) If all subgroups are normal in G then G is abelian

(d) If G is abelian then all subgroups of G are normal

Soln. For option (a)

Take $G = S_3$

Let $x = (123)$ and $y = (12)$

Clearly $o(x) = 3$ and $o(y) = 2$

$xy = (123)(12) = (1)(23)$

$\Rightarrow o(xy) = 2$

$\Rightarrow$ Option (a) is incorrect

For option (b)

Let $G = (P(\mathbb{N}), \Delta)$

G is an infinite group in which every element is of finite order.

For option (c)

Let $G = Q_8$

All subgroups of G are normal in G but G is non abelian.

For option (d) :

We know that if G is abelian group then all subgroup of G are normal

Correct option is (d).

6. Let $GL_n(\mathbb{R})$ denote the group of all $n \times n$ matrices with real entries (with respect to matrix multiplication) which are invertible. Pick out the normal subgroups from the following: **[NBHM-2010]**

(a) The subgroup of all real orthogonal matrices

(b) The subgroup of all invertible diagonal matrices

(c) The subgroup of all matrices with determinant equal to unity

Soln. For option (a)

Take $n = 2$

Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

Clearly $A \in GL_2(\mathbb{R})$ is an orthogonal matrix.

Let $B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}$

Let $BAB^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$

Clearly BAB^{-1} is not an orthogonal matrix

$\Rightarrow$ The subgroup of all real orthogonal matrices does not form a normal subgroup of $GL_n(\mathbb{R})$

For option (b):

Take $n = 2$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Clearly A is an invertible diagonal matrix.

Let $B = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{R})$

Consider $BAB^{-1} = \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ 0 & 2 \end{bmatrix}$

$\Rightarrow$ BAB^{-1} is not a diagonal matrix

$\Rightarrow$ The subgroup of all invertible diagonal matrices does not form a normal subgroup.

For option (c)

Let $A \in GL_n(\mathbb{R})$ be such that det (A) = 1

Let $B \in GL_n(\mathbb{R})$

Consider det $\left(BAB^{-1}\right) = \det(B)\det(A)\det\left(B^{-1}\right)$

$= \det(B)\det(A)\dfrac{1}{\det(B)} = \det(A) = 1$

$\Rightarrow \det\left(BAB^{-1}\right) = 1$

Thus the subgroup of all matrices with determinant equal to unity form a normal subgroup of $GL_n(\mathbb{R})$

Hence correct option is (c)

7. Let G be the group of invertible upper triangular matrices in $\mathbb{M}_2(\mathbb{R})$. If we write $A \in G$ as $A = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$, which of the following define a normal subgroup of G? **[NBHM-2014]**

(a) $H = \{A \in G \mid a_{11} = 1\}$

(b) $H = \{A \in G \mid a_{11} = a_{22}\}$

(c) $H = \{A \in G \mid a_{11} = a_{22} = 1\}$

Soln. Given $G = \left\{\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \in M_2(\mathbb{R}) \middle| a_{11}a_{22} \neq 0\right\}$

For option (a)

Given $H = \{A \in G \mid a_{11} = 1\}$

Let $B \in G$

Let $A \in H \Rightarrow A = \begin{pmatrix} 1 & a'_{12} \\ 0 & a'_{22} \end{pmatrix}; a'_{22} \neq 0$

Let $B \in G$

$\Rightarrow$ $B = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}; a_{11}a_{22} \neq 0$

Consider $BAB^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}\begin{pmatrix} 1 & a'_{12} \\ 0 & a'_{22} \end{pmatrix}\dfrac{1}{a_{11}a_{22}}\begin{pmatrix} a_{22} & -a_{12} \\ 0 & a_{11} \end{pmatrix}$

$$= \frac{1}{a_{11}a_{22}} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} a_{22} & -a_{12} + a_{11}a'_{12} \\ 0 & a_{11}a'_{22} \end{pmatrix}$$

$$= \frac{1}{a_{11}a_{22}} \begin{pmatrix} a_{11}a_{22} & -a_{12}a_{11} + a_{11}^2 a'_{12} + a_{12}a_{11}a'_{22} \\ 0 & a_{11}a_{22}a'_{22} \end{pmatrix} \in H$$

$\Rightarrow H$ is normal in G

For option (b)

Given $H = \{A \in G \mid a_{11} = a_{22}\}$

Let $A \in H \Rightarrow A = \begin{pmatrix} a'_{11} & a'_{12} \\ 0 & a'_{11} \end{pmatrix}; a'_{11} \neq 0$

Let $B \in G$

$\Rightarrow B = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}; a_{11}a_{22} \neq 0$

Consider $BAB^{-1} = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} a'_{11} & a'_{12} \\ 0 & a'_{11} \end{pmatrix} \frac{1}{a_{11}a_{22}} \begin{pmatrix} a_{22} & -a_{12} \\ 0 & a_{11} \end{pmatrix}$

$$= \frac{1}{a_{11}a_{22}} \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix} \begin{pmatrix} a'_{11}a_{22} & -a'_{11}a_{12} + a'_{12}a_{11} \\ 0 & a_{11}a'_{11} \end{pmatrix}$$

$$= \frac{1}{a_{11}a_{22}} \begin{pmatrix} a_{11}a'_{11}a_{22} & -a_{11}a'_{11}a_{12} + a'_{12}a_{11}^2 + a_{12}a_{11}a'_{11} \\ 0 & a_{11}a'_{11}a_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a'_{11} & \dfrac{a'_{12}a_{11}}{a_{22}} \\ 0 & a'_{11} \end{pmatrix} \in H$$

$\Rightarrow H$ is normal in G

Similarly we can prove for option (c)

Hence correct option is (a), (b) and (c)

8. For real numbers a and b, define the mapping $\tau_{a,b} : \mathbb{R} \to \mathbb{R}$ by $\tau_{a,b}(x) = ax + b$. Let

$G = \{\tau_{a,b} : a, b \in \mathbb{R}, a \neq 0\}$

Under composition of mappings, this becomes a group. Which of the following subgroups of G are normal ?

(a) $H = \{\tau_{a,b} \mid a \neq 0, a \in \mathbb{Q}, b \in \mathbb{R}\}$ **[NBHM-2015]**

(b) $H = \{\tau_{1,b} \mid b \in \mathbb{R}\}$

(c) $H = \{\tau_{1,b} \mid b \in \mathbb{Q}\}$

Soln. Given $G = \{\tau_{a,b} : a, b \in \mathbb{R}, a \neq 0\}$

For option (a):

Given $H = \{\tau_{a,b} \mid a \neq 0, a \in \mathbb{Q}, b \in \mathbb{R}\}$

Let $x \in G \Rightarrow x = \tau_{a,b}$ for some $a, b \in \mathbb{R}, a \neq 0$

Let $h \in H \Rightarrow h = \tau_{a',b'}$ for some $a' \in \mathbb{Q}, b' \in \mathbb{R}, a' \neq 0$

Consider $xhx^{-1} = \tau_{(a,b)}\tau_{(a',b')}\tau_{(a,b)}^{-1}$

$$\tau_{(a,b)}\tau_{(a',b')}\tau_{(a,b)}^{-1}(y)=\tau_{(a,b)}\left(a'\left(\frac{y}{a}-\frac{b}{a}\right)+b'\right)$$

$$=aa'\left(\frac{y}{a}-\frac{b}{a}\right)+ab'+b$$

$$=a'(y-b)+ab'+b$$

$$=a'y+ab'-a'b+b$$

$$=\tau_{(a',\,ab'-a'b+b)}(y)\in H$$

$\Rightarrow H$ is normal in G.

For option (b)

Given $H=\{\tau_{1,b}\mid b\in\mathbb{R}\}$

Let $x\in G$ and $h\in H$

$\Rightarrow x=\tau_{a,b}$ for some $a,b\in\mathbb{R}$, $a\neq 0$

and $h=\tau_{1,b'}$ for some $b'\in\mathbb{R}$

Consider $\tau_{a,b}\tau_{1,b'}\tau_{\left(\frac{1}{a},\frac{-b}{a}\right)}(y)=\tau_{a,b}\tau_{1,b'}\left(\frac{1}{a}y-\frac{b}{a}\right)$

$$=\tau_{a,b}\left(\frac{1}{a}y-\frac{b}{a}+b'\right)$$

$$=a\left(\frac{1}{a}y-\frac{b}{a}+b'\right)+b$$

$$=y-b+ab'+b$$

$$=\tau_{1,ab'}(y)$$

$\Rightarrow \tau_{1,ab'}\in H$

$\Rightarrow H$ is normal subgroup of G.

For option (c):

Given $H=\{\tau_{1,b}\mid b\in\mathbb{Q}\}$

Let $x\in G$ and $h\in H$

$\Rightarrow x=\tau_{a,b}$ for some $a,b\in\mathbb{R}, a\neq 0$

and $h=\tau_{1,b'}$ for some $b'\in\mathbb{Q}$

Now $\tau_{a,b}\tau_{1,b'}\tau_{\frac{1}{a},\frac{-b}{a}}=\tau_{1,ab'}$

If $a\in\mathbb{R}-\mathbb{Q}$ then $ab'\notin\mathbb{Q}$

$\Rightarrow \tau_{1,ab'}\notin H$

$\Rightarrow H$ is not normal in G.

Hence correct option is (a) and (b)

9. Let $n\in\mathbb{N}, n\geq 2$. Which of the following subgroups are normal in $GL_n(\mathbb{C})$? **[NBHM-2017]**

(a) $H=\{A\in GL_n(\mathbb{C})\mid A \text{ is upper triangular}\}$

(b) $H=\{A\in GL_n(\mathbb{C})\mid A \text{ is diagonal}\}$

(c) $H=\{A\in GL_n(\mathbb{C})\mid \det(A)=1\}$

Soln. Given $G = GL_n(\mathbb{C})$

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

Consider $BAB^{-1} = \begin{pmatrix} 1 & -1 \\ +1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -2 & 2 \end{pmatrix}$

$= \frac{1}{2}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$

Clearly BAB^{-1} is not an upper triangular matrix

$\Rightarrow H = \{A \in GL_n(\mathbb{C}) \mid A$ is upper triangular matrix$\}$ and $H = \{A \in GL_n(\mathbb{C}) \mid A$ is diagonal$\}$ are not normal in G.

$H = \{A \in GL_n(\mathbb{C}) \mid \det(A) = 1\}$ is normal in G. $(\because \det(BAB^{-1}) = \det A)$

Hence correct option is (c)

10. Let G be an abelian group with the identity e. Which one of the following statements are true.

(a) $H = \{x \in G \mid \text{order of } x \text{ is odd}\}$ is a subgroup of G **[H.C.U. 2018]**

(b) $H = \{x \in G \mid \text{order of } x \text{ is even}\} \cup \{e\}$ is a subgroup of G

(c) Every subgroup of G is normal

(d) G is cyclic

Soln. For option (a)

Given $H = \{x \in G \mid \text{order of } x \text{ is odd}\}$

To prove that H is a subgroup of G, it is sufficient to prove that H is closed

Let $a, b \in H$

$\Rightarrow o(a) =$ an odd integer and $o(b) =$ an odd integer

We know that $o(ab)$ divides lcm of $o(a)$ and $o(b)$ in an abelian group.

$\Rightarrow$ $o(ab)$ is an odd integer

$\Rightarrow$ $ab \in H$

$\Rightarrow$ H is a subgroup of G.

For option (b)

Take $G = \mathbb{Z}_{20}$

H = The set of all even ordered elements in $\mathbb{Z}_{20} \cup \{e\} = \{0, 1, 2, 5, ...\}$

H is not a subgroup of G. ($\because$ 1+ 5 = 6 does not belong to H)

Hence correct option is (a) and (c).

11. Any normal subgroup of order 2 is contained in the center of the group. **[TIFR-2013]**

Soln. Let G be a group and H be any subgroup of G of order 2. i.e. $H = \{e, x\}$

We have to prove $x \in Z(G)$ i.e. $xy = yx \ \forall y \in G$

Let $y \in G$ be any arbitrary element.

Since H is normal in G.

$\Rightarrow yxy^{-1} \in H$

$\Rightarrow$ Either $yxy^{-1} = e$ or $yxy^{-1} = x$

If $yxy^{-1} = e \Rightarrow x = e$

$\Rightarrow yxy^{-1} \neq e$

$\Rightarrow y\,x\,y^{-1} = x$

$\Rightarrow yx = xy \ \forall\ y \in G$

$\Rightarrow x \in Z(G)$

$\Rightarrow H \subseteq Z(G)$

12. In each of the following, state whether the given set is a normal subgroup or, is a subgroup which is not normal or, is not a subgroup of $GL_n(\mathbb{C})$.

(a) The set of matrices with determinant equal to unity

(b) The set of invertible upper triangular matrices

(c) The set of invertible matrices whose trace is zero **[NBHM-2012]**

Soln. For option (a).

We know that the set of all matrices with determinant equal to unity form a normal subgroup.

For option (b):

The set of all invertible upper triangular matrices form a subgroup of $GL_n(\mathbb{C})$ but not a normal subgroup.

For option (c)

Let $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$

Clearly A and B are invertible matrices whose trace is zero

Now $AB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\text{Trace}(AB) = 2 \neq 0$

$\Rightarrow$ The set of invertible matrices whose trace is zero does not form a subgroup of G.

13. Let A and B be normal subgroups of a group G. Suppose $A \cap B = \{e\}$, where e is the unit element of the group G, then

(a) for any $a \in A$ and $b \in B$, $ab = ba$.

(b) for only some $a \in A$ and $b \in B$, $ab = ab$.

(c) $ab \neq ba$ for any $a \in A$ and $b \in B$.

(d) $ab \neq ba$ for some $a \in A$ and $b \in B$.

Soln. Since $\forall a \in A, b \in B$

$(aba^{-1})b^{-1} \in B \ \&\ a(bab^{-1}) \in A$

as $A \cap B = \{e\} \Rightarrow aba^{-1}b^{-1} = e$

$\Rightarrow ab = ba \forall a \in A, b \in B$

option (a) is correct

14. Which of the following is true ?

(a) $\exists$ two subgroups H,K, which are not normal but HK is a subgroup

(b) $\nexists$ two subgroups H,K, which are not normal but HK is a subgroup

(c) If $|H|$ = 14 and |K| = 33 then $|H \cap K|$ is greater than 1

(d) A_5 has subgroup of order 15

Soln. Let $G = S_4, H = \{I,(12)\}$

$K = \{I,(123),(132)\}$

Hence $HK = \{I,(12),(123),(132),(12)(123),(12)(132)\}$

$= \{I,(12),(123),(23),(13),(132)\}$

$KH = \{I,(12),(123),(132),(23),(13)\}$

Thus HK = KH $\Rightarrow$ HK is subgroup

but H and K are not normal subgroup of G

$\therefore$ option (a) is correct & (b) is false

Since $|H \cap K|$ must divide $|H| = 14$ and $|k| = 33$

$\Rightarrow |H \cap K| = 1$

$\therefore$ option (c) is false

If A_5 has subgroup of order 15. Then A_5 must have an element of order 15 as group of order 15, is cycle but A_5 does not have an element of order 15.

$\therefore$ option (d) is false

Correct option is (a)

15. Let G be a group and H, K be subgroups of G such that $G = H \oplus K$. Let N be a normal subgroup of G such that $N \cap H = \{e\}$ and $N \cap K = \{e\}$. Then N is

(a) abelian (b) Non abelian (c) Cyclic (d) None of these

Soln. Since $G = H \times K$, H and K are normal subgroup of G.

Now $\forall n \in N$, $h \in H$, $k \in K$, $nh = nh$ and $nk = kn$ $\left\{\begin{array}{l}\text{if H, k be normal subgroup of G \& H} \cap \text{K} = \{e\} \\ \text{then hk} = kh \ \forall h \in H \text{ and k} \in \text{K}\end{array}\right\}$

Let $a, b \in N$, then $\exists h \in H, k \in K$ such that $b = hk$.

Now $ab = a(hk) = (ah)k = (ha)k = h(ak)$

$= h(ka) = (hk)a = ba$

$\Rightarrow$ $ab = ba$

$\Rightarrow$ N is abelian

$\therefore$ **option (a) is correct**

16. Let G be a group and $H = \{g^2 \mid g \in G\}$. Then

(a) H must be normal subgroup
(b) H is sub group but need not to be normal subgroup
(c) H is not sub group of G.
(d) H may not be subgroup and if it is a subgroup then it must be normal.

Soln. Suppose $G = A_4$ contains all twelve even permutation of S_4 which are {I, (12)(34), (13)(24),(14)(23)} and the 8 3-cycles elements. Since $I^2 = I$, $((ab)\,(cd))^2$ = I and square of any 3-cycle is a 3-cycle, we notice H will contains I and 8-3 element cycles so that $o(H) = 9$ and $9 \nmid 12$. So H can not be subgroup of G. Option (a), (b) are false

Suppose now H is subgroup then if $h \in H, g \in G$ be any elements, then

$g^{-1} \in G \Rightarrow g^{-2} \in H$ also $gh \in G$

$\Rightarrow (gh)^2 \in H$

$\Rightarrow g^{-2}(gh)(gh) \in H$

$\Rightarrow g^{-1}hg \in H$

$\therefore$ H is normal in G.

$\therefore$ **Correct option is (d)**