Chapter 7

Wronskian

1.10 (1) LINEAR DEPENDENCE OF SOLUTIONS

Consider the initial value problem consisting of the homogeneous linear equation

$$y'' + py' + qy = 0 \quad ...(1)$$

with variable co-efficients $p(x)$ and $q(x)$ and two initial conditions $y(x_0) = k_0.,\ y'(x_0) = k_1$ ...(2)

Lets its general solution be $y = c_1 y_1 + c_2 y_2$ ...(3)

which is made up of two linearly dependent solutions y_1 and y_2*.

If $p(x)$ and $q(x)$ are continuous functions on some open interval I and x_0 is any fixed point on I, then the above initial value problem has a **unique solution** $y(x)$ on the interval I.

(2) **Theorem.** If $p(x)$ and $q(x)$ are continuous on an open interval I, then the solutions y_1 and y_2 of (1) are linearly dependent in I if and only if the Wronskian† $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$ for some x_0 on I. If there is an $x = x_1$ in I at which $W(y_1, y_2) \neq 0$, then y_1, y_2 are linearly independent on I.

Proof : If y_1, y_2 are linearly dependent solutions of (1) then there exist two constants c_1, c_2 not both zero, such that $c_1 y_1 + c_2 y_2 = 0$ ...(4)

Differentiating w.r.t. x, $c_1 y_1' + c_2 y_2' = 0$ ...(5)

Eliminating c_1, c_2 from (4) and (5), we get

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = 0$$

Conversely, suppose $W(y_1, y_2) = 0$ for some $x = x_0$ on I and show that y_1, y_2 are linearly dependent.

Consider the equation

$$\left.\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) = 0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0 \end{aligned}\right\} \quad ...(6)$$

which, on eliminating c_1, c_2 gives $W(y_1, y_2) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0$

Hence the system has a solution in which c_1, c_2 are not both zero.

Now introduce the function $\bar{y}(x) = c_1 y_1(x) + c_2 y_2(x)$.

Then $y(x)$ is a solution of (1) on I. By (6), this solution satisfies the initial conditions $y(x_0)=0$ and $y'(x_0)=0$. Also since $p(x)$ and $q(x)$ are continuous on I, this solution must be unique. But $y \equiv 0$ is obviously another solution of (1) satisfying the given initial conditions. Hence $\bar{y} \equiv y$ i.e. $c_1 y_1 + c_2 y_2 \equiv 0$ in I. Now since c_1, c_2 are not both zero, it implies that y_1 and y_2 are linearly dependent on I.

Remark : (1) Let f and g be two differentiable function on an interval I and

$W\left(f(x), g(x)\right) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} \neq 0$ for some $x \in I$ then $f(x)$ and $g(x)$ are linearly indepedent function.

(2) Converse of (1) is not true for example, $f(x) = x|x|$ and $g(x) = x^2$ are two linearly independent solution and $W(f, g) = 0 \ \forall \ x \in \mathbb{R}$

(3) If $f(x)$ and $g(x)$ are linearly dependent function then

$$W\left(f(x), g(x)\right) = \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = 0 \ \forall x \in I$$

(4) Converse of (3) is not true for example, $f(x) = x|x|$ and $g(x) = x^2, W(f, g) = 0 \ \forall \ x \in \mathbb{R}$ and $f(x)$ and $g(x)$ are two linearly independent solution.

(5) Let y_1 and y_2 are two solution of an ODE. Then

y_1 and y_2 are L.I $\Leftrightarrow W(y_1, y_2) \neq 0 \ \forall x$

y_1 and y_2 are L.D $\Leftrightarrow W(y_1, y_2) = 0 \ \forall x$

ABEL'S THEOREM

Let $a_1, a_2, a_3, \ldots, a_n$ be continuous functions on an interval I containing the point x_0.

Let $\phi_1, \phi_2, \phi_3, \ldots, \phi_n$ be n solution of ODE, $y^{(n)} + a_1(x) y^{(n-1)} + a_2(x) y^{(n-2)} + \ldots + a_{n-1} y' + a_n y = 0$.

Then wronskin of solution $\phi_1, \phi_2, \ldots, \phi_n$ is $W(x) = W(x_0) e^{-\int_{x_0}^{x} a_1(t)dt}$

Also $W(x) = ce^{-\int a_1(x)dx}$ where c is constant.

Example-1

Show that the two functions $\sin 2x$, $\cos 2x$ are independent solutions of $y'' + 4y = 0$.

Soln. Substituting $y_1 = \sin 2x$ (or $y_2 = \cos 2x$) in the given equation we find that y_1, y_2 are its solutions.

Also $W(y_1, y_2) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2 \neq 0$

for any value of x. Hence the solutions y_1, y_2 are linearly independent.

Previous Year Solved Problems

Example-2

Consider the following statements regarding the two solutions $y_1(x) = \sin x$ and $y_2(x) = \cos x$ of $y'' + y = 0$.

(i) They are linearly dependent solutions of $y'' + y = 0$ **[D.U. 2015]**

(ii) Their wronskian is 1

(iii) They are linearly independent solutions of $y'' + y = 0$

which of the statements is true?

(a) (i) and (ii)
(b) (ii) and (iii)
(c) (iii)
(d) (i)

Soln. $y_1 = \sin, y_2 = \cos(x)$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \sin x & \cos(x) \\ \cos(x) & -\sin x \end{vmatrix} = -1$$

$\therefore w \neq 0 \Rightarrow$ y_1 and y_2 are linearly independent

Statement (iii) is only true statement

$\therefore$ **Option (c) is Correct**

Example-3

Let $y_1(x)$ and $y_2(x)$ be two solutions of $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + (\sec x)y = 0$ with Wronskian $W(x)$. If $y_1(0) = 1, \left(\frac{dy_1}{dx}\right)_{x=0} = 0$ and $W\left(\frac{1}{2}\right) = \frac{1}{3}$, then $\left(\frac{dy_2}{dx}\right)_{x=0}$ equals **[GATE-2006]**

(a) $\frac{1}{4}$
(b) 1
(c) $\frac{3}{4}$
(d) $\frac{4}{3}$

Soln. $(1+x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \sec xy = 0 \Rightarrow \frac{d^2y}{dx^2} - \frac{2x}{1-x^2}\frac{dy}{dx} + \sec xy = 0$

By Abel's theorem, $y'' + p(x)y' + Q(x) = 0$

$$W(x) = c.e^{-\int p(x)dx} = c.e^{\int \frac{2x}{1-x^2}dx} = c.e^{-\int \frac{2x}{x^2-1}dx}$$

$$W(x) = c.e^{-\log|x^2-1|} = \frac{c}{x^2-1}$$

$$W(x) = \frac{c}{x^2-1}$$

Since $W\left(\frac{1}{2}\right) = \frac{1}{3} \Rightarrow \frac{1}{3} = \frac{c}{\frac{1}{4}-1}$

$$\Rightarrow \frac{1}{3} = \frac{-4}{3}c \Rightarrow c = \frac{-1}{4}$$

So, $W(x) = \frac{-1}{4(x^2-1)} = \frac{1}{4(1-x^2)}$

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \Rightarrow W(0) = \begin{vmatrix} y_1(0) & y_2(0) \\ y_1'(0) & y_2'(0) \end{vmatrix} \Rightarrow \frac{1}{4} = \begin{vmatrix} 1 & y_2(0) \\ 0 & y_2'(0) \end{vmatrix}$$

$$\Rightarrow \frac{1}{4} = y_2'(0) \Rightarrow \left(\frac{dy_2}{dx}\right)_{x=0} = \frac{1}{4}$$

Option (a) is Correct

Example-4

Given below four sets $\{f_1, f_2, f_3\}$ of functions defined on $\mathbb{R}$. Determine which set is linearly dependent

(a) $\{f_1(x) = x^2, f_2(x) = x^4, f_3(x) = x^{-2}\}$ (b) $\{f_1(x) = x, f_2(x) = x+1, f_3(x) = x+2\}$

(c) $\{f_1(x) = \cos x, f_2(x) = \sin x, f_3(x) = 1\}$ (d) $\{f_1(x) = e^x, f_2(x) = e^{-x}, f_3(x) = 1\}$ **[CUCET- 2016]**

Soln. $W(f_1, f_2, f_3)(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ f_1' & f_2'(x) & f_3'(x) \\ f_1''(x) & f_2''(x) & f_3''(x) \end{vmatrix}$

$$f_1(x) = x^2, f_2(x) = x^4, f_3(x) = \frac{1}{x^2}$$

$$W(x) = \begin{vmatrix} x^2 & x^4 & \frac{1}{x^2} \\ 2x & 4x^3 & \frac{-2}{x^3} \\ 2 & 12x^2 & \frac{6}{x^4} \end{vmatrix} = x^2 \times \left(\frac{24}{x} + \frac{24}{x}\right) - x^4\left(\frac{12}{x^3} + \frac{4}{x^3}\right) + \frac{1}{x^2}\left(2yx^3 - 8x^3\right)$$

$= 48x - 16x + 16x = 48x$ which is non zero for some $x \in \mathbb{R}$

$\therefore \{f_1, f_2, f_3\}$ is linearly independent set

Also, $f_1(x) = x, f_2(x) = x+1, f_3(x) = x+2$

$$W(x) = \begin{vmatrix} x & x+1 & x+2 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0 \; \forall x \in \mathbb{R}$$

$W(x)=0\ \ \forall x\in\mathbb{R}$ but we cannot say anything

Also, there exists $a=-1, b=2, c=-1$ such that

$a\ f_1(x)+bf_2(x)+cf_3(x)=0$.

Therefore, $\{f_1, f_2, f_3\}$ is linearly dependent

Now, $f_1(x)=\cos x, f_2(x)=\sin x,\ f_3(x)=1$

$$W(x)=\begin{vmatrix}\cos x & \sin x & 1\\ -\sin x & \cos x & 0\\ -\cos x & -\sin x & 0\end{vmatrix}=1\neq 0\ \forall\ x\in\mathbb{R}$$

Therefore, $\{f_1, f_2, f_3\}$ is linearly independnet set on $\mathbb{R}$

again, $f_1(x)=e^x,\ f_2(x)=e^{-x},\ f_3(x)=1$

$$W(x)=\begin{vmatrix}e^x & e^{-x} & 1\\ e^x & -e^{-x} & 0\\ e^x & e^{-x} & 0\end{vmatrix}=2\neq 0\ \forall x\in\mathbb{R}$$

Therefore, $\{f_1, f_2\ f_3\}$ is linearly independnet on $\mathbb{R}$.

Option (b) is Correct .

***Example*-5**

Let f_1 and f_2 be two solutions of $a_0(x)\dfrac{d^2y}{dx^2}+a_1(x)\dfrac{dy}{dx}+a_2(x)y=0$, where a_0, a_1 and a_2 are continuous on $[0,1]$ and $a_0(x)\neq 0$ for all $x\in[0,1]$. Moreover, let $f_1\left(\dfrac{1}{2}\right)=f_2\left(\dfrac{1}{2}\right)=0$. Then **[D.U. 2016]**

(a) one of f_1 and f_2 must be identically zero (b) $f_1(x)=f_2(x)$ for all $x\in[0,1]$

(c) $f_1(x)=c\ f_2(x)$ for some constant c (d) none of these

Soln. $a_0(x)\dfrac{d^2y}{dx^2}+a_4(x)\dfrac{dy}{dx}+a_2(x)y=0 \Rightarrow \dfrac{d^2y}{dx^2}+\dfrac{a_1(x)}{a_0(x)}\dfrac{dy}{dx}+\dfrac{a_2(x)}{a_0(x)}y=0$

We have, $f_1\left(\dfrac{1}{2}\right)=0,\ f_2\left(\dfrac{1}{2}\right)=0$

$$W(x) = \begin{vmatrix} f_1(x) & f_2(x) \\ f_1'(x) & f_2'(x) \end{vmatrix}$$

$$W\left(\frac{1}{2}\right) = \begin{vmatrix} f_1\left(\frac{1}{2}\right) & f_2\left(\frac{1}{2}\right) \\ f_1'\left(\frac{1}{2}\right) & f_2'\left(\frac{1}{2}\right) \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ f_1'\left(\frac{1}{2}\right) & f_2'\left(\frac{1}{2}\right) \end{vmatrix} = 0$$

We know, f_1 and f_2 are solution of ODE than f_1 and f_2 are linearly dependent iff $W(x) = 0$ for some x.

Since $W(x) = 0$ for $x = \frac{1}{2}$

Therefore, $f_1(x)$ and $f_2(x)$ are linearly dependent solution of the given ODE

$\Rightarrow$ $f_1(x) = cf_2(x)$ for some constant c.

Option (c) is Correct

***Example*-6**

For which of the following pair of functions $y_1(x)$ and $y_2(x)$, continuous functions $p(x)$ and $q(x)$ can be determined on [–1, 1] such that $y_1(x)$ and $y_2(x)$ give two linearly independent solutions of

$y'' + p(x)\,y' + q(x)\,y = 0,\ x \in [-1, 1]$ **[GATE-2007]**

(a) $y_1(x) = x\sin(x),\ y_2(x) = \cos(x)$

(b) $y_1(x) = x\,e^x,\ y_2(x) = \sin(x)$

(c) $y_1(x) = e^{x-1},\ y_2(x) = e^x - 1$

(d) $y_1(x) = x^2,\ y_2(x) = \cos(x)$

Soln. y_1 and y_2 are two linearly independent solution of given ODE if and only if $w(x) \neq 0$ for all x.

$$w(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$y_1 = x\sin x,\ y_2 = \cos x$

$$w(x) = \begin{vmatrix} x\sin x & \cos x \\ x\cos x + \sin x & -\sin x \end{vmatrix}$$

$= -x\sin^2 x - x\cos^2 x - \cos x\sin x = -x - \cos x\sin x$ which is zero for $x = 0$

$w(x) = 0$ for $x = 0 \Rightarrow y_1$ & y_2 are linearly dependent again, $y_1 = xe^x$, $y_1 = \sin x$

$$w(x) = \begin{vmatrix} xe^x & \sin x \\ (x+1)e^x & \cos x \end{vmatrix} = xe^x \cos x - xe^x \sin x + e^x \sin x$$

$w(0) = 0 \Rightarrow y_1$ & y_2 are linearly dependent

Also, $y_1 = e^{x-1}, y_2 = e^x - 1$

$$w(x) = \begin{vmatrix} e^{x-1} & e^x - 1 \\ e^{x-1} & e^x \end{vmatrix} = e^{2x-1} - e^{2x-1} + e^{x-1} = e^{x-1}$$ which is non-zero for all $x \in \mathbb{R}$.

y_1 & y_2 are linearly independenet.

Also, $y_1 = x^2, y_2 = \cos(x)$

$$w(x) = \begin{vmatrix} x^2 & \cos(x) \\ 2x & -\sin x \end{vmatrix} = -x^2 \sin x - 2x \cos x$$

$w(0) = 0 \Rightarrow y_1$ & y_2 are linearly dependenet

Option (c) is Correct