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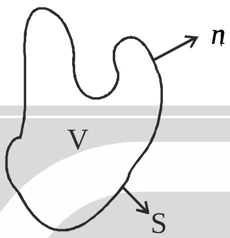
Gauss's Divergence Theorem

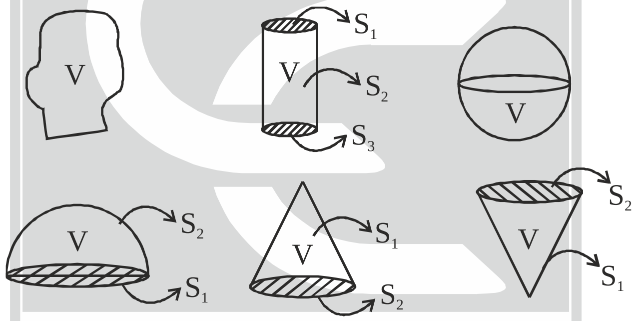
Gauss's Divergencev Theorem

Theorem: Let *V* be the volume bounded by a closed piecewise smooth, simple surface *S* oriented outward. If $\vec{F} = f(x,y,z)\hat{i} + g(x,y,z)\hat{j} + h(x,y,z)\hat{k}$, where *f*, *g* and *h* have continuous first partial derivatives on some open set containing *V* and if $\hat{n}$ is the outward unit normal on S, then

$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV}$$



Note : (i) To apply Gauss Theorem volume *V* must be closed region.



(ii) If $\vec{F} = f\hat{i} + g\hat{j} + h\hat{k}$, then *f*, *g*, and *h* must continuous first partial derivation in *V*.

Note : (i) $\iint_S \vec{F}\cdot\hat{n}\,dS = \iiint_V div\vec{F}\,dV$, $\hat{n}$ outward unit normal on *S*

(ii) $\iint_S \vec{F}\cdot(-\hat{n})\,dS = \iiint_V div\vec{F}\,dV$, $\hat{n}$ inward unit normal on *S*

$\Rightarrow \iint_S \vec{F}\cdot\hat{n}\,dS = -\iiint_V div\vec{F}\,dV$

(iii) $\iint_S f\,dydz + g\,dxdz + h\,dxdy = \iiint_V \nabla\cdot\vec{F}\,dV$

(iv) $\iint_S f\cos\alpha + g\cos\beta + h\cos\gamma)\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$,

where outer unit normal $\hat{n} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$ on *S*.

(v) $\iint_{S_1} \vec{F}\cdot\hat{n}\,dS + \iint_{S_2} \vec{F}\cdot\hat{n}\,dS + \iint_{S_3} \vec{F}\cdot\hat{n}\,dS + \ldots + \iint_{S_n} \vec{F}\cdot\hat{n}\,dS = \iiint_V \nabla\cdot\vec{F}\,dV$

Where volume *V* is a closed region bounded by surfaces $S_1, S_2, S_3, \ldots\ldots\ldots S_n$.

Ex.1: Let $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$ and S is the surface of sphere $x^2 + y^2 + z^2 = a^2$. Then find the flux through S.

Soln. We have,

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

Since, x, y and z are having continuous first partial derivative. Therefore, we get,

$$div\,\vec{F} = \nabla \cdot \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1 + 1 + 1 = 3$$

$$\therefore \quad Flux = \iint_S \vec{F} \cdot \hat{n} dS$$

Since sphere is a closed volume so by the Gauss Theorem,

$$\iiint_V div\vec{F}dv = \iiint_V 3dv = 3 \times \text{volume of sphere of radius} = 3 \times \frac{4\pi}{3} a^3 = 4\pi a^3$$

Ex.2: Let $\vec{F} = x^2\hat{i} + z^2\hat{k}$ and S the surface of the box $|x| \le 1, |y| \le 3$ and $0 \le z \le 2$. Then evaluate $\iint \vec{F} \cdot \hat{n} dS$, where $\hat{n}$ is inward unit normal on S.

Soln. We have,

$$\vec{F} = x^2\hat{i} + z^2\hat{k} \Rightarrow \nabla \cdot \vec{F} = 2x + 2z$$

$\hat{n}_1 = -\hat{n}$ (outward unit normal on S)

Therefore, by Gauss Theorem, (Since volume is closed)

$$\iint_S \vec{F} \cdot \hat{n} dS = \iiint_V \nabla \cdot \vec{F} dV \Rightarrow \iint_S \vec{F} \cdot (-\hat{n} dS) = \iiint_V 2(x+z)dxdydz$$

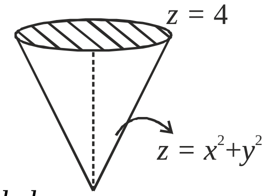
$$\Rightarrow \quad \iint_S \vec{F} \cdot \hat{n} dS = -\int_0^2 \int_{-3}^3 \int_{-1}^1 2(x+z)dxdydz = -2\int_0^2 \int_{-3}^3 \int_{-1}^1 zdxdydz = -2\int_0^2 zdz \times \int_{-3}^3 dy \times \int_{-1}^1 dx = -4 \times 6 \times 2 = -48$$

Ex.3. Compute $\iint_S x^2 dydz + y^2 dxdz + z^2 dxdy$, where S is surface bounded by $z = x^2 + y^2$ and $z = 4$ by Gauss Theorem and then directly solving the surface integral

Soln. We have,

$$\iint_S x^2 dydz + y^2 dxdz + z^2 dxdy = \iint (x^2\hat{i} + y^2\hat{j} + z^2\hat{k}) \cdot d\vec{S} = \iint \vec{F} \cdot \hat{n} dS$$

Here $\vec{F} = x^2\hat{i} + y^2\hat{j} + z^2\hat{k}$



$\Rightarrow \quad \nabla \cdot \vec{F} = 2x + 2y + 2z = 2(x + y + z)$

By Gauss Theorem (since V is closed volume)

$$\iint_S \vec{F} \cdot \hat{n} dS = \iiint_V \nabla \cdot \vec{F} dV = \iint_R \int_{x^2+y^2}^{4} 2(x+y+z)dzdydx$$

$$= 2\iint_R \left((x+y)(4 - x^2 - y^2) + \left(8 - \frac{(x^2+y^2)^2}{2} \right) \right) dxdy$$ (R = Projecton on surface on xy-plane $(x^2 + y^2 \le 4)$)

$$= 2\int_0^{2\pi} \int_0^2 \left(r^2(\cos\theta + \sin\theta)(4 - r^2) + \left(8r - \frac{r^5}{2} \right) \right) drd\theta = 2\int_0^{2\pi} \int_0^2 \left(8r - \frac{r^5}{2} \right) drd\theta = 4\pi \left[4r^2 - \frac{r^6}{12} \Big|_0^2 \right] = \frac{128}{3}\pi$$

Now, $\iint_S \vec{F} \cdot \hat{n} dS = \iint_{S_1} \vec{F} \cdot \hat{n} dS + \iint_{S_2} \vec{F} \cdot \hat{n} dS$

$\hat{n} = \dfrac{\nabla(x^2+y^2-z)}{|\nabla(x^2+y^2-z)|}$ on S_1 away from the axis of the cone

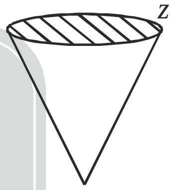
$= \dfrac{2x\hat{i}+2y\hat{j}-\hat{k}}{\sqrt{4x^2+4y^2+1}}$ and $dS = \dfrac{dxdy}{|\hat{n}\cdot\hat{k}|} = \sqrt{4x^2+4y^2+1}\,dxdy$

$\hat{n} = \hat{k}$ on S_2

$$\therefore \iint_{S_1} \vec{F}\cdot\hat{n}dS = \iint_{R:x^2+y^2\le 4} \frac{2x^3+2y^3-z^2}{\sqrt{4x^2+4y^2+1}}(\sqrt{4x^2+4y^2+1})dxdy = \iint_{x^2+y^2\le 4}\left(2x^3+2y^3\right)dxdy - \iint_{x^2+y^2\le 4} z^2dxdy$$

$$= 2\int_0^2\int_0^{2\pi} r^4(\cos^3\theta+\sin^3\theta)drd\theta - \iint_{x^2+y^2\le 4}(x^2+y^2)^2dxdy$$

$$= 0 - \int_0^{2\pi}\int_0^2 r^5 drd\theta = -2\pi\times\frac{32}{3} = \frac{-64\pi}{3}$$



and $\iint_{S_2}\vec{F}\cdot\hat{n}dS = \iint_{x^2+y^2\le 4}\vec{F}\cdot\hat{k}\,dS$

$$= \iint 16dxdy = 16\times 4\pi = 64\pi$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}dS = \iint_{S_1}\vec{F}\cdot\hat{n}dS + \iint_{S_2}\vec{F}\cdot\hat{n}dS = -\frac{64\pi}{3} + 64\pi = 64\pi\left(\frac{3-1}{3}\right) = \frac{128\pi}{3}$$

Ex.4: Evaluate $\iint_S (x^2+y^2+z^2)dS$ where $S:\ x^2+y^2+z^2=1$, by Gauss Theorem and verify it by solving the surface integral directly

Soln. We have $\iint (x^2+y^2+z^2)dS = \iint_{x^2+y^2\le 1}(x\hat{i}+y\hat{j}+z\hat{k})\cdot(x\hat{i}+y\hat{j}+z\hat{k})dS = \iint \vec{F}\cdot\hat{n}dS$

Here, $\vec{F} = x\hat{i}+y\hat{j}+z\hat{k}$

$\hat{n} = x\hat{i}+y\hat{j}+z\hat{k}$ outward unit normal on S

$\therefore\ \nabla\cdot\vec{F} = 1+1+1 = 3$

Therefore, By Gauss Theorem.

$$\iint_S \vec{F}\cdot\hat{n}dS = \iiint_V \nabla\cdot\vec{F}dV = \iiint_V 3dV = 3\times\text{volume of sphere of radius } 1 = 4\pi$$

Now, $\iint_S (x^2+y^2+z^2)dS = \iint_{x^2+y^2\le 1}(x^2+y^2+z^2)\dfrac{dxdy}{|z|}$

$$= \iint_{\substack{x^2+y^2\le 1\\ z>0}} \frac{x^2+y^2+1-x^2-y^2}{\sqrt{1-x^2-y^2}}dxdy - \iint_{\substack{x^2+y^2\le 1\\ z<0}} \frac{x^2+y^2+1-x^2-y^2}{-\sqrt{1-x^2-y^2}}dxdy$$

$$= 2\iint_{x^2+y^2\le 1}\frac{1}{\sqrt{1-x^2-y^2}}dxdy = 2\int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{1-r^2}}drd\theta = \left.\frac{(1-r^2)}{\left(-\frac{1}{2}\right)}\right|_0^1 \times 2\pi = 4\pi$$

$\therefore$ Gauss theorem is varified

Ex.5: Evaluate $\iint_S \frac{(a^2x^2+b^2y^2+c^2z^2)}{\sqrt{a^4x^2+b^4y^2+c^4z^2}}dS$, where S is the surface of $a^2x^2+b^2y^2+c^2z^2=1$, $(a,b,c)>0$ and verify Gauss theorem.

Soln. We have, $\iint_S \frac{a^2x^2+b^2y^2+c^2z^2}{\sqrt{a^4x^2+b^4y^2+c^4z^2}}dS = \iint_S x\hat{i}+y\hat{j}+z\hat{k}\cdot\frac{a^2x\hat{i}+b^2y\hat{j}+c^2z\hat{k}}{\sqrt{a^4x^2+b^4y^2+c^4z^2}} = \iint_S \vec{F}\cdot\hat{n}dS$

Here, $\vec{F}=x\hat{i}+y\hat{j}+z\hat{k}$, $\hat{n}=\frac{a^2x\hat{i}+b^2y\hat{j}+c^2z\hat{k}}{\sqrt{a^4x^2+b^2y^2+c^2z^2}}$ outward unit normal on S.

Since $\vec{F}$ has continuous partial derivative and V is closed. Then By Gauss Theorem,

$$\iint_S \vec{F}\cdot\hat{n}dS = \iiint_V \nabla\cdot\vec{F}dV = \iiint 3dV = 3\times \text{volume of ellipsoid} = 3\times\frac{4\pi}{3}\left(\frac{1}{a}\right)\left(\frac{1}{b}\right)\left(\frac{1}{c}\right) = \frac{4\pi}{abc}$$

Now, $\iint_S \vec{F}\cdot\hat{n}dS = \iint_R \frac{a^2x^2+b^2y^2+c^2z^2}{\sqrt{a^4x^2+b^4y^2+c^4z^2}}\frac{dxdy}{|\hat{n}\cdot\hat{k}|} = \iiint_{a^2x^2+b^2y^2\le 1}\frac{(a^2x^2+b^2y^2+c^2z^2)}{|c^2z|}dxdy$

$$= \iint_{\substack{a^2x^2+b^2y^2\le 1\\ z>0}}\frac{a^2x^2+b^2x^2+c^2z^2}{c^2z}dxdy + \iint_{\substack{a^2x^2+b^2y^2\le 1\\ z<0}}\frac{a^2x^2+b^2x^2+c^2z^2}{-c^2z}dxdy$$

$$= \iint_{\substack{a^2x^2+b^2y^2\le 1\\ z>0}}\frac{a^2x^2+b^2x^2+1-a^2x^2-b^2y^2}{c^2\,\frac{\sqrt{1-a^2x^2-b^2y^2}}{c}}dxdy + \iint_{\substack{a^2x^2+b^2y^2\le 1\\ z<0}}\frac{a^2x^2+b^2x^2+c^2z^2}{c^2\,\frac{\sqrt{1-a^2x^2-b^2y^2}}{c}}dxdy$$

$$= \frac{2}{c}\iint_{\substack{a^2x^2+b^2y^2\le 1\\ z>0}}\frac{1}{\sqrt{1-a^2x^2-b^2y^2}}dxdy = \frac{2}{abc}\iint_{x^2+y^2\le 1}\frac{1}{\sqrt{1-u^2-v^2}}dudv, \text{ Put } x=\frac{u}{a},\ y=\frac{v}{b}$$

$$= \frac{1}{abc}\int_0^{2\pi}\int_0^1 \frac{2r}{\sqrt{1-r^2}}drd\theta = \frac{2\pi}{abc}\times\left(\left.\frac{-(1-r^2)^{1/2}}{\left(\frac{1}{2}\right)}\right|_0^1\right) = \frac{4\pi}{abc}$$

$\therefore$ Gauss theorem gets varified

Ex.6: Let $\vec{F}=xy\hat{i}+yz\hat{j}+xz\hat{k}$ and S is surface bounded by $0\le x\le 1$, $0\le y\le 1, 0\le z\le 1$. Then evaluate $\iint_S \vec{F}\cdot\hat{n}dS$ and verify the gauss theorem

Soln. We have,

$$\vec{F}=xy\hat{i}+yz\hat{j}+xz\hat{k} \Rightarrow \nabla\cdot\vec{F}=y+z+x$$

Since V is closed by S, then By Gauss Theorem,

$$\iint_S \vec{F}\cdot\hat{n}dS = \iiint_V \nabla\cdot\vec{F}dS = \int_0^1\int_0^1\int_0^1 (x+y+z)dxdydz$$

$$= \int_0^1\int_0^1\int_0^1 xdxdydz + \int_0^1\int_0^1\int_0^1 ydydxdz + \int_0^1\int_0^1\int_0^1 zdzdxdy = \frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{3}{2}$$

Now, $\iint \vec{F}\cdot\hat{n}dS = \iint_{S_1}\vec{F}\cdot\hat{n}dS + \iint_{S_2}\vec{F}\cdot\hat{n}dS + \iint_{S_3}\vec{F}\cdot\hat{n}dS + \iint_{S_4}\vec{F}\cdot\hat{n}dS + \iint_{S_5}\vec{F}\cdot\hat{n}dS + \iint_{S_6}\vec{F}\cdot\hat{n}dS$

$$= \iint_{x=0\,\text{plane}} \vec{F}\cdot(-\hat{i})dS + \iint_{y=0}\vec{F}\cdot(-\hat{j})dS + \iint_{z=0}\vec{F}\cdot(-\hat{k})dS + \iint_{x=1}\vec{F}\cdot(\hat{i})dS + \iint_{y=1}\vec{F}\cdot(\hat{j})dS + \iint_{z=1}\vec{F}\cdot(\hat{k})dS$$

(Be carefull $\hat{n}$ is outward normal on each surface).

$$= \iint_{x=0}(yz)\hat{j}\cdot(-\hat{i})dydz + \iint_{y=0}(xz)\hat{k}\cdot(-\hat{j})dxdz + \iint_{z=0}(xy)\hat{i}\cdot(-\hat{k})dxdy + \iint_{x=1}(y\hat{i}+yz\hat{j}+z\hat{k})\cdot(\hat{i})dydz$$

$$+ \iint_{y=1}(x\hat{i}+z\hat{j}+xz\hat{k})\cdot(\hat{j})dxdz + \iint_{z=1}(xy\hat{i}+y\hat{j}+x\hat{k})\cdot dxdy$$

$$= 0+0+0+\int_0^1\int_0^1 ydydz + \int_0^1\int_0^1 zdydz + \int_0^1\int_0^1 xdxdy = \frac{1}{2}+\frac{1}{2}+\frac{1}{2} = \frac{3}{2}$$

Thus, Gauss Theorem is varified

Application:

Suppose (as in previous example) $S = S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5$ and we evaluate $\iint_S \vec{F}\cdot\hat{n}dS$, where $\hat{n}$ unit normal to S,

If we include S_6 surface i.e. $\sigma = S \cup S_6$,then region defined by σ is a closed volume and we can apply Gauss Theorem,

Therefore, $\iint_\sigma \vec{F}\cdot\hat{n}dS = \iiint_V \nabla\cdot\vec{F}dV \Rightarrow \iint_S \vec{F}\cdot\hat{n}dS + \iint_{S_6}\vec{F}\cdot\hat{n}dS = \frac{3}{2}$

$$\Rightarrow \iint_S \vec{F}\cdot\hat{n}dS + \frac{1}{2} = \frac{3}{2} \Rightarrow \iint_S \vec{F}\cdot\hat{n}dS = \frac{3}{2}-\frac{1}{2} = 1$$

Ex.7: If S is the surface of the sphere $x^2+y^2+z^2=1$, then the value of the integral $\iint_S (ax\,dy\,dz + by\,dz\,dx + cz\,dx\,dy)$ is

(a) $\pi(a+b+c)$ (b) $\frac{4}{3}(a+b+c)$ (c) $\frac{4}{3}\pi(a+b+c)$ (d) $\frac{4}{3}\pi abc$ **[B.H.U.-2011, 2014]**

Soln. We have, $x^2+y^2+z^2=1$

$$\iint_S ax\,dydz + by\,dzdx + cz\,dxdy = \iint_S \left(ax\hat{i}+by\hat{j}+cz\hat{k}\right)\cdot d\vec{S}$$

Since S is closed surface then By Gauss Divergence Theorem,

$$= \iiint \nabla\cdot\left(ax\hat{i}+by\hat{j}+cz\hat{k}\right)dV = \iiint (a+b+c)\,dV = (a+b+c)\times\text{volume of sphere} = \frac{4}{3}\pi(a+b+c)$$

Ex.8: If S is any closed surface enclosing a volume V and $F = xi + 2yj + 3zk$, then $\iint_S F\cdot\hat{n}\,dS$ is

(a) $2V$ (b) $3V$ (c) $4V$ (d) $6V$ **[B.H.U.-2012]**

Soln. We have,

$\vec{F} = x\hat{i} + y\hat{j} + 2z\hat{k} \Rightarrow \nabla\cdot\vec{F} = 1+1+2 = 4$

Therefore, Since S is closed surface then by Gauss Divergence Theorem,

$$\iint \vec{F}\cdot\hat{n}dS = \iiint \nabla\cdot\vec{F}dV = \iiint 4dV = 4V$$

Ex.9. The value of $\iint_S (xdydz + ydzdx + zdxdy)$, where S is the surface of the sphere $x^2 + y^2 + z^2 = a^2$ is

(a) $2\pi a^3$ (b) $4\pi a^3$ (c) $4\pi a^2$ (d) $2\pi a$ **[ISM-2017]**

Soln. Since S is a closed surface then by Gauss divergence theorem,

$$\iint_S x\,dydz + y\,dzdx + z\,dxdy = \iint_S \left(x\hat{i} + y\hat{j} + z\hat{k}\right)\cdot d\vec{S}$$

$$= \iiint \nabla\cdot\left(x\hat{i} + y\hat{j} + z\hat{k}\right)dV = \iiint 3dV = 3\times \text{volume of sphere} = 3\times\frac{4\pi}{3}a^3 = 4\pi a^3$$

Ex.10: Let S be the surface of the cylinder $x^2 + y^2 = 4$ bounded by the planes $z = 0$ and $z = 1$. Then the surface integral $\iint_S ((x^2 - x)\hat{i} - 2xy\hat{j} + z\hat{k})\cdot\hat{n}dS$ **[HCU-2012]**

(a) –1 (b) 0 (c) 1 (d) None of (a), (b), (c)

Soln. We have, $\vec{F} = \left(x^2 - x\right)\hat{i} - 2xy\hat{j} + z\hat{k} \Rightarrow \nabla\cdot\vec{F} = 2x - 1 - 2x + 1 = 0$

Since S be closed surface then by Gauss Divergence Theorem,

$$\iint_s\left(\left(x^2 - x\right)\hat{i} - 2xy\hat{j} + z\hat{k}\right)\cdot\hat{n}dS = \iiint \nabla\cdot\vec{F}\,dV = \iiint 0\,dV = 0$$

Ex.11: Let S be the sphere with center at the origin and radius 1. Let $\overline{f}$ is a vector field given by $\overline{f}(x, y, z) = (z - 2xyz)\hat{i} + 9x^2yz^2\hat{j} + (yz^2 - 3x^2z^3)\hat{k}$. If $\hat{n}$ is the outward normal then, the value of $\iint_S \overline{f}\cdot\hat{n}dS =$ **[H.C.U.-2013]**

(a) 0 (b) $\frac{4}{3}\pi$ (c) π (d) $\frac{4}{3}\pi^3$

Soln. We have,

$x^2 + y^2 + z^2 = 1$ and $\vec{f} = \left(z - 2xyz\right)\hat{i} + 9x^2yz^2\hat{j} + \left(yz^2 - 3x^2z^3\right)\hat{k}$

$$\Rightarrow \nabla\cdot\vec{F} = -2zy + 9x^2z^2 + 2yz - 9x^2z^2 = 0 \therefore \iint_S \vec{f}\cdot\hat{n}dS = \iiint_V \vec{\nabla}\cdot\vec{f}\,dV = 0$$

Ex.12: Let $\overline{f} = (1, f_2(x, y, z), f_3(x, y, z))$ be solenoidal field where f_2, f_3 are scalar valued functions. Let S be the unit sphere in $\mathbb{R}^3$ and $\hat{n}$ be unit outward normal. Then $\int_S x\overline{f}\cdot\hat{n}dS =$ **[H.C.U.-2011]**

(a) 0 (b) π (c) $4\pi/3$ (d) 4π

Soln. We have,

Since $\vec{f}$ is solenoidal, then $\nabla\cdot\vec{f} = 0$

Therefore $\nabla\cdot x\vec{f} = x\nabla\cdot\vec{f} + \vec{f}\cdot\nabla(x) = 0 + (1, f_2, f_3)\cdot(1, 0, 0) = 1$

Since S is a closed surface then by Gauss divergence Theorem,

$$\iint x\vec{f}\cdot\hat{n}dS = \iiint \nabla\cdot\left(x\vec{f}\right)dV = \iiint dV = \text{Volume of sphere of radius } 1 = \frac{4\pi}{3}$$

Since S is closed surface then by Gauss divergence Theorem,

$$\iint_S \vec{f}\cdot\hat{n}dS = \iiint \nabla\cdot\vec{f}\,dV = \iiint 0\,dV = 0$$

Ex.13: Let B be the unit sphere in $\mathbb{R}^3$. The value of $\iint_B (x^2+2y^2-3z^2)dS$ is **[H.C.U.-2014]**

(a) 4π (b) $\frac{4}{3}\pi$ (c) 6π (d) none of the above

Soln. We have, $x^2+y^2+z^2=1$

$$\hat{n}=\frac{\nabla\phi}{|\nabla\phi|}=x\hat{i}+y\hat{j}+z\hat{k}$$

$$\left(x^2+2y^2-3z^2\right)dS=\left(x\hat{j}+2y\hat{j}-3z\hat{k}\right)\cdot\left(x\hat{i}+y\hat{j}+z\hat{k}\right)dS=\left(x\hat{j}+2y\hat{j}-3z\hat{k}\right)\cdot\hat{n}dS$$

Let $\vec{f}=x\hat{i}+2y\hat{j}-3z\hat{k}\Rightarrow\nabla\cdot\vec{f}=1+2-3=0$

Since S is a closed surface, $\therefore$ by Gauss divergence theorem,

$$\iint_B\left(x^2+2y^2-3z^2\right)dS=\iint_B\vec{f}\cdot\hat{n}dS=\iiint\nabla\cdot\vec{f}\,dV=\iiint 0\,dV=0$$

Ex.14: Let V be the region which is common to the solid sphere $x^2+y^2+z^2\le 1$ and the solid cylinder $x^2+y^2\le 0.5$. Let ∂V be the boundary of V and $\hat{n}$ be the unit outward normal drawn at the boundary. Let $\vec{F}=(y^2+z^2)\hat{i}+(z^2-2x^2)\hat{j}+(x^2+2y^2)\hat{k}$. Then the value of $\iint_{\partial V}\vec{F}\cdot\hat{n}dS$ is equal to **[H.C.U.-2016]**

(a) 0 (b) 1 (c) –1 (d) π

Soln. $\vec{F}=\left(y^2+z^2\right)\hat{i}+\left(z^2-2x^2\right)\hat{j}+\left(x^2+2y^2\right)\hat{k}\Rightarrow\nabla.\vec{F}=0+0+0=0$

Since ∂V is a closed surface so by Gauss Divergence Theorem,

$$\iint_{\partial v}\vec{F}\cdot\hat{n}dS=\iiint\nabla\cdot\vec{f}\,dv=\iiint 0dV=0$$

Ex.15: The value of the surface integral $\iint_s\vec{F}\cdot\hat{n}dS$, where S is the surface of the sphere $x^2+y^2+z^2=4$, n is the unit outward normal and $\vec{F}=x\hat{i}+y\hat{j}+z\hat{k}$, is **[GATE-1999]**

(a) 32π (b) 16π (c) 8π (d) 64π

Soln. We have, $\vec{F}=x\hat{i}+y\hat{j}+z\hat{k}\Rightarrow\ \nabla\cdot\vec{F}=1+1+1=3$

Since S is a closed surface $\therefore$ by the Gauss Divergence theorem

$$\iint_S\vec{F}\cdot\hat{n}dS=\iiint\nabla\cdot\vec{F}dV=\iiint 3dV=3\times\text{volume of the sphere of radius }2=3\times\frac{4\pi}{3}\times 8=32\pi$$

Ex.16: Let V be the volume of a region bounded by a smooth closed surface S. Let r denote the position vector and n denote the outward unit normal to S. Then the integral $\iint_s\vec{r}\cdot\hat{n}\,dS$ equals. **[GATE-2002]**

(a) V (b) $\frac{V}{3}$ (c) $3V$ (d) 0

Soln. We have,

$$\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}\Rightarrow\nabla\cdot\vec{r}=1+1+1=3$$

Since S is a closed surface. So by Gauss Divergence Theorem

$$\iint\vec{r}\cdot\hat{n}dS=\iiint\nabla\cdot\vec{r}dV=\iiint 3dV=3V$$

Ex.17: Let $B=\{(x,y,z)\mid x,y,z\in\mathbb{R} \text{ and } x^2+y^2+z^2\le 4\}$ Let $v(x,y,z)=x\hat{i}+y\hat{j}+z\hat{k}$ be a vector- valued function on B. If $r^2=x^2+y^2+z^2$, the value of the integral $\iiint_B \nabla\cdot\left(r^2v(x,y,z)\right)dV$ is **[GATE-2003]**

(a) 16π (b) 32π (c) 64π (d) 128π

Soln. We have,

$$\phi=x^2+y^2+z^2-4 \Rightarrow \nabla\phi=2x\hat{i}+2y\hat{j}+2z\hat{k},\ \hat{n}=\frac{\nabla\phi}{|\nabla\phi|}=\frac{x\hat{i}+y\hat{j}+z\hat{k}}{2}$$

S is a closed surface. So by the Gauss Divergence Theorem,

$$\iiint_B \nabla\cdot((r^2v))dV=\iint_S r^2v\cdot\hat{n}dS=\iint r^2(x\hat{i}+y\hat{j}+z\hat{k}).\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{2}dS$$

$$=\iint\frac{r^2(x^2+y^2+z^2)}{2}dS=\iint\frac{(x^2+y^2+z^2)^2}{2}\frac{dxdy}{|\hat{n}\cdot\hat{k}|}$$

$$=\iint(x^2+y^2+z^2)^2\frac{dxdy}{|z|}=2\iint(x^2+y^2+z^2)\frac{dxdy}{z}$$

$$=2\iint(x^2+y^2+4-x^2-y^2)^2\frac{dxdy}{\sqrt{4-x^2-y^2}}$$

$$=32\iint\frac{dxdy}{\sqrt{4-x^2-y^2}}=32\int_0^{2\pi}\int_0^2\frac{rdrd\theta}{\sqrt{4-r^2}}$$

$$=32\pi\int_0^2\frac{2rdr}{\sqrt{4-r^2}}=32\pi\times\left.\left(\frac{-(4-r^2)^{1/2}}{\left(\frac{1}{2}\right)}\right)\right|_0^2=32\pi\times2\times2=128\pi$$

Alternative solution

We have,

$$\nabla\cdot(r^2v)=r^2\nabla\cdot v+v\cdot\nabla r^2=3r^2+2r^2=5r^2$$

Therefore,

$$\iiint\nabla\cdot(r^2v)dV=\iiint 5r^2dV=5\int_0^{2\pi}\int_0^{\pi}\int_0^2 r^2\cdot r^2\sin\phi drd\phi d\theta=5\int_0^{2\pi}d\theta\times\int_0^{\pi}\sin\phi d\phi\times\int_0^2 r^2dr=5\times2\pi\times2\times\frac{32}{5}=128\pi$$

Ex.18: Let S be the surface bounding the region $x^2+y^2\le1, x\ge0, |z|\le1$, and $\hat{n}$ tbe the unit outer normal to S. Then $\iint_S\left[\left(\sin^2 x\right)i+2y\hat{j}-z\left(1+\sin 2x\right)\hat{k}\right]\cdot\hat{n}dS$ equals **[GATE-2004]**

(a) 1 (b) $\frac{\pi}{2}$ (c) π (d) 2π

Soln. We have,

$$\vec{F}=\sin^2 x\hat{i}+2y\hat{j}-z(1+\sin2x)\hat{k} \Rightarrow \nabla\cdot\vec{F}=\sin2x+2-1-\sin2x=1$$

Since S is a closed surface. So by Gauss divergence Theorem,

$$\iint_s\left[(\sin^2 x)\hat{i}+2y\hat{j}-z(1+\sin2x)\hat{k}\right]\cdot\hat{n}dS=\iiint\nabla\cdot\vec{F}dV=\iiint1\cdot dV=\iiint dV=\frac{\pi\times1\times2}{2}=\pi$$

Ex.19. Let $W=\{(x,y,z)\in\mathbb{R}^3 : 1\le x^2+y^2+z^2\le 4\}$ and $F:W\to\mathbb{R}^3$ be defined by

$F(x,y,z)=\dfrac{(x,y,z)}{\left[x^2+y^2+z^2\right]^{3/2}}$ for $(x,y,z)\in W$. If ∂W denotes the boundary of W oriented by the outward nomal n to W, then $\iint_{\partial W} F\cdot n dS$ is equal to **[GATE-2008]**

(a) 0 (b) 4π (c) 8π (d) 12π

Soln. We have,

$$\vec{F}(x,y,z)=\frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}=\frac{\vec{r}}{r^3}$$

$$\Rightarrow\quad \nabla\cdot F=\nabla\cdot\frac{\vec{r}}{r^3}=\frac{3}{r^3}-\frac{3}{r^4}\times\frac{\vec{r}}{r}\cdot\vec{r}=\frac{3}{r^3}-\frac{3}{r^3}=0$$

Since F is differentiable in W and ∂W is a closed surface. So by Gauss divergence Theorem.

$$\iint_{\partial W}\vec{F}\cdot\hat{n}dS=\iiint_W \nabla\cdot\vec{F}dV=\iiint 0dV=0$$

Ex.20. The flux of the vector field $\vec{u}=x\hat{i}+y\hat{j}+z\hat{k}$ flowing out through the surface of the ellipsoid

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1,\ a>b>0$ is **[GATE-2012]**

(a) πabc (b) $2\pi abc$ (c) $3\pi abc$ (d) $4\pi abc$

Soln. We have,

$$\vec{u}=x\hat{i}+y\hat{j}+z\hat{k}\Rightarrow\ \nabla\cdot\vec{u}=1+1+1=3$$

Therefore,

$$\text{Flux}=\iint_S \vec{u}\cdot d\vec{S}$$

Since S is a closed surface. So by Gauss Divergence Theorem,

$$\text{Flux}=\iiint_V \nabla\cdot\vec{u}dV=3\iiint dV=3\times\text{volume of ellipsoid}=3\times\frac{4\pi}{3}abc=4\pi abc$$

Ex.21: Consider the unit sphere $S=\{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2=1\}$ and the unit normal vector $\hat{n}=(x,y,z)$ at each point (x,y,z) on S. The value of the surface integral **[GATE-2015]**

$$\iint_S\left\{\left(\frac{2x}{\pi}+\sin(y^2)\right)x+\left(e^z-\frac{y}{\pi}\right)y+\left(\frac{2z}{\pi}+\sin^2 y\right)z\right\}d\sigma$$ is equal to _______

Soln. We have, $\phi=x^2+y^2+z^2-1,\hat{n}=\dfrac{\nabla\phi}{|\nabla\phi|}=x\hat{i}+y\hat{j}+z\hat{k}$

$$\iint_S\left\{\left(\frac{2x}{\pi}+\sin(y^2)\right)x+\left(e^z-\frac{y}{\pi}\right)y+\left(\frac{2z}{\pi}+\sin^2 y\right)z\right\}dS$$

$$=\iint\left\{\left(\frac{2x}{\pi}+\sin(y^2)\right)\hat{i}+\left(e^z-\frac{y}{\pi}\right)\hat{j}+\left(\frac{2z}{\pi}+\sin^2 y\right)\hat{k}\right\}\cdot\hat{n}dS$$

Let $\vec{F} = \left(\frac{2x}{\pi} + \sin y^2\right)\hat{i} + \left(e^z - \frac{y}{\pi}\right)\hat{j} + \left(\frac{2z}{\pi} + \sin^2 y\right)\hat{k}$

$\Rightarrow \quad \nabla \cdot \vec{F} = \frac{2}{\pi} - \frac{1}{\pi} + \frac{2}{\pi} = \frac{3}{\pi}$

Since S is a closed surface, so by Gauss Divergence Theorem,

$$\iint_S \left\{\left(\frac{2x}{\pi} + \sin(y^2)\right)\hat{i} + \left(e^z - \frac{y}{\pi}\right)\hat{j} + \left(\frac{2z}{\pi} + \sin^2 y\right)\hat{k}\right\} \cdot \hat{n}dS = \iiint_V \nabla \cdot \vec{F} dV = \iiint \frac{3}{\pi} dV$$

$$= \frac{3}{\pi} \times \text{Volume of sphere of radius } 1 = \frac{3}{\pi} \times \frac{4\pi}{3} = 4$$

Ex.24: Let S be the surface of the solid $V = \{(x, y, z) : 0 \le x \le 1, 0 \le y \le 2, 0 \le z \le 3\}$ Let $\hat{n}$ denote the unit outward normal to S and let $\vec{F}(x, y, z) = x\hat{i} + y\hat{j} + z\hat{k},\ (x, y, z) \in V$ Then the surface integral $\iint_s \vec{F} \cdot \hat{n}\, dS$ equal ______. **[GATE-2018]**

Soln. We have, $\vec{F} = x\hat{i} + y\hat{j} + z\hat{k} \Rightarrow \nabla \cdot \vec{F} = 1 + 1 + 1 = 3$

Since S is closed surface, so by Gauss Divergence Theorem,

$$\iint_S \vec{F} \cdot \hat{n}dS = \iiint \nabla \cdot \vec{F} dV = \iiint 3dV = 3 \times \text{volume of the cuboid} = 3 \times 1 \times 2 \times 3 = 18$$

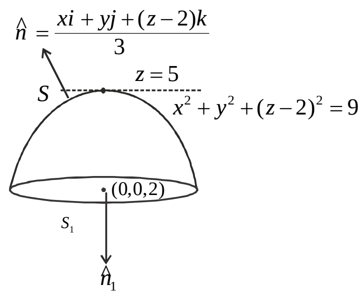
Ex.25: Consider the hemisphere $x^2 + y^2 + (z-2)^2 = 9, 2 \le z \le 5$ and the vector field $\vec{F}(x, y, z) = x\hat{i} + y\hat{j} + (z-2)\hat{k}$. Then surface ingegral $\iint (\vec{F} \cdot \vec{n})\, d\sigma$ evaluated over the hemisphere with $\vec{n}$ denoting the unit outward normal, is **[GATE-2006]**

(a) 9π (b) 27π (c) 54π (d) 162π

Soln. We have, $\vec{F} = x\hat{i} + y\hat{j} + (z-2)\hat{k}, \vec{F} = x\hat{i} + y\hat{j}\ \ at\ z = 2 \Rightarrow \nabla \cdot \vec{F} = 1 + 1 + 1 = 3$

Therefore,

$$\iint_S \vec{F} \cdot \hat{n}dS + \iint_{S_1} (x\hat{i} + y\hat{j}) \cdot (-\hat{k})dS = \iiint \nabla \cdot \vec{F} dV$$



$$\Rightarrow \iint_S \vec{F} \cdot \hat{n}dS + \iint (x\hat{i} + y\hat{j}).(-\hat{k})dS = 3\iiint dV$$

$$\Rightarrow \iint \vec{F} \cdot \hat{n}dS = 3 \times \text{volume of the hemisphere}$$

$$= 3 \times \frac{4\pi}{3 \times 2} \cdot 27 = 54\pi$$

EXERCISE

1. Let $\vec{F} = x\hat{i} + 2y\hat{j} + 3z\hat{k}$, S be the surface of the sphere $x^2 + y^2 + z^2 = 1$ and $\hat{n}$ be the inward unit normal vector to S. Then $\oint_S \vec{F}\cdot\hat{n}dS$ is equal to

2. Let S be a closed surface for which $\iint_S \vec{r}\cdot\hat{n}dS = 1$. Then the volume enclosed by the surface is

3. The value of integral $\oint_S \vec{F}\cdot\hat{n}dS$, where $\vec{F} = 3x\hat{i} + 2y\hat{j} + z\hat{k}$ and S is the closed surface, given by the planes $x = 0,\ x = 1,\ y = 0,\ y = 2,\ z = 0$ and $z = 3$ is

4. For any closed surface S, the surface integral $\oint \text{curl}\,\vec{F}\cdot\hat{n}dS$ is equal to

5. $\vec{F} = (2x+5z)\hat{i} - (x^2z + y)\hat{j} + (y^2 + 2z)\hat{k}$, then value of integral $\oint_S \vec{F}\cdot\hat{n}dS$ where S is the surface of sphere having centre at (2, 3, 1) and radius a is equal to

6. If S be any closed surface enclosing a volume V and $\vec{F} = 2x\hat{i} + 3y\hat{j} + 7z\hat{k}$. Then, the value of surface integral $\oint \vec{F}\cdot\hat{n}dS$ is equal to

7. If $\vec{F} = \nabla\phi$ and $\nabla^2\phi = 0$, show that for a closed surfaces $\oint \phi\vec{F}\cdot\hat{n}dS = \int_V F^2 dV$.

8. Verify divergence theorem for $\vec{F} = 4xz\hat{i} - y^2\hat{j} + 4z\hat{k}$ taken over the cube bounded by $x = 0, y = 0, z = 0, x = a, y = a, z = a$.

9. Suppose $\vec{A} = 6z\hat{i} + (2x+y)\hat{j} - x\hat{k}$. Evaluate $\iint_S \vec{A}\cdot\hat{n}dS$ over the entire surface S of the region bounded by the cylinder $x^2 + z^2 = 9,\ x = 0,\ y = 0,\ z = 0,$ and $y = 8$.

10. Evaluate $\iint_S \vec{r}\cdot\hat{n}dS$ over.

(a) the surface S of the unit cube bounded by the coordinate planes and the planes $x = 1,\ y = 1,\ z = 1$;

(b) the surface of a sphere of radius a with center at (0, 0, 0).

11. Suppose $\vec{A} = 4xz\hat{i} + xyz^2\hat{j} + 3z\hat{k}$. Evaluate $\iint_S \vec{A}\cdot\hat{n}dS$ over the entire surface of the region above the xy-plane bounded by the cone $z^2 = x^2 + y^2$ and the plane $z = 4$.

12. Suppose $\vec{F} = (2x^2 - 3z)\hat{i} - 2xy\hat{j} - 4x\hat{k}$. Evaluate

(a) $\iiint_V \nabla\cdot\vec{F}dV$ and (b) $\iiint_V \nabla\times\vec{F}dV$, where V is the closed region bounded by the planes $x = 0, y = 0, z = 0,$ and $2x + 2y + z = 4$.

13. Verify the divergence theorem for $\vec{A} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ taken over the region in the first octant bounded by $y^2 + z^2 = 9$ and $x = 2$.

14. Evaluate $\iint_S \hat{r}\cdot\hat{n}dS$ where (a) S is the sphere of radius 2 with center at (0, 0, 0)

(b) S is the surface of the cube bounded by $x = -1, y = -1, z = -1, x = 1, y = 1, z = 1$

(c) S is the surface bounded by the paraboloid $z = 4 - (x^2 + y^2)$ and the *xy*-plane.

15. Suppose S is any closed surface enclosing a volume V and $\vec{A} = ax\hat{i} + by\hat{j} + cz\hat{k}$. Prove that $\iint_S \vec{A}\cdot\hat{n}dS = (a+b+c)V$.

16. Let $\vec{F} = x^2\hat{i} - z^2\hat{k}$ and S the surface of the box $|x| \le 1, |y| \le 3$ and $0 \le z \le 2$. Then evaluate the surface integral $\iint \vec{F}\cdot\hat{n}dS$

17. Let $\vec{F} = e^x\hat{i} + e^y\hat{j} + e^z\hat{k}$ and *S* the surface of cube $|x| \le 1, |y| \le 1, |z| \le 1$. Then find the flux through surface of cube.

18. Let $\vec{F} = (x^3 - y^3)\hat{i} + (y^3 - z^3)\hat{j} + (z^3 - x^3)\hat{k}$ and *S* the surface of sphere $x^2 + y^2 + z^2 \le 25, z \ge 0$. Then fined upward flux through part of $x^2 + y^2 + z^2 = 25$.

19. Let $\vec{F} = \sin y\hat{i} + \cos x\hat{j} + \cos z\hat{k}$ and *S* the surface bounded by $x^2 + y^2 = 4, z = \pm 2$. Then evaluate $\iint_S \vec{F}\cdot\hat{n}dS$

20. Let $\vec{F} = 2x^2\hat{i} + \frac{1}{2}y^2\hat{j} + \sin\pi z\hat{k}$ and S to the surface of tetrhedron with vertices (0, 0, 0), (1,0,0),(0,1,0) (0, 0, 1). Then evaluate $\iint_S \vec{F}\cdot\hat{n}dS$

21. Let $\vec{F} = x^2\hat{i} + y\hat{j} + z^2\hat{k}$ and *S* be the surface of cone $x^2 + y^2 \le z^2, 0 \le z \le h$. Then evaluate $\iint_S \vec{F}\cdot\hat{n}dS$

22. Let $\vec{F} = xy\hat{i} + yz\hat{j} + xz\hat{k}$ and *S* is the surface of the cone $x^2 + y^2 \le 4z^2, 0 \le z \le 2$. Then find the value flux through *S*.

23. Let $\vec{F} = (x^2 + y)\hat{i} + z^2\hat{j} + (e^y - z)\hat{k}$ and *S* is the surface of the rectangular solid bounded by the co-ordinate planes and the planes *x* = 3, *y* = 1 and *z* = 2. Then find the flux of $\vec{F}$ across the surface *S* with outward oriantation

24. Evaluate $\iint_S (xz^3 - yx^3 + y^3z)dS$, where *S* is surface of sphere $x^2 + y^2 + z^2 = a^2$

25. Evaluate $\iint_S ((x-z)x + (y-x)y)dS$ where S is the surface of cylinder of $x^2 + y^2 = 1$ between *z* = 0 to z =1.

26. Let $\vec{F} = (x-z)\hat{i} + (y-x)\hat{j} + (z-y)\hat{k}$ and *S* is the surface of the cylindrical solid bounded by $x^2 + y^2 = a^2$, z = 0 and z = 1. Then find the flux across the surface *S* with outward orientation

27. Let $\vec{F} = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$ and *S* by the surface of the cylindrical solid bounded by $x^2 + y^2 = 4, z = 0$ and $z = 3$. Then evaluate $\iint_S \vec{F}\cdot\hat{n}dS$

28. If $\vec{F}(x,y,z)=(x^3-e^y)\hat{i}+(y^3+\sin z)\hat{j}+z^3-xy)\hat{k}$, where σ is the surface of the solid bounded by $z=\sqrt{4-x^2-y^2}$ and the xy-plane, then find outward flux of $\vec{F}$ across σ.

29. If $\vec{F}(x,y,z)=2x^2z\hat{i}+y^2\hat{j}+z^2\hat{k}$ where σ is the surface of the conical solid bounded by $z=\sqrt{x^2+y^2}$ and $z=1$, then find outward flux of $\vec{F}$ across σ.

30. If $\vec{F}(x,y,z)=x^3\hat{i}+x^2y\hat{j}+xy\hat{k};\sigma$ is the surface of the solid bounded by $z=4-x^2, y+z=5, z=0$, and $y=0$, then find outward flux of $\vec{F}$ across σ.

Prove that from Q. 31 to 35

31. $\iint_{\sigma} \text{curl}\,\vec{F}\cdot\hat{n}\,dS=0$

32. $\iint \nabla f\cdot\hat{n}\,dS=\iiint_G \nabla^2 f\,dV \quad \left(\nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}\right)$

33. $\iint_{\sigma}(f\nabla g)\cdot\hat{n}\,dS=\iiint_G (f\nabla^2 g+\nabla f\cdot\nabla g)\,dV$

34. $\iint_{\sigma}(f\nabla g-g\nabla f)\cdot\hat{n}\,dS=\iiint_G (f\nabla^2 g-g\nabla^2 f)\,dV$

35. $\iint_{\sigma}(f\,\hat{n}).\vec{v}\,dS=\iiint_G \nabla f\cdot\vec{v}\,dV \quad$ ($\vec{v}$ a fixed vector)

36. Find all positive values of k such that $\vec{F}(r)=\frac{r}{\|\vec{r}\|^2}$

37. Let $\vec{F}=(y-x)\hat{i}+(z-y)\hat{j}+(y-x)\hat{k}$ and D: The cube bounded by the planes $x=\pm1, y=\pm1$, and $z=\pm1$. Then find the outward flux $\vec{F}$ across the boundary of the region D.

38. Let $\vec{F}=y\hat{i}+xy\hat{j}-z\hat{k}$, and D: The region inside the solid cylinder $x^2+y^2\le 4$ between the plane z = 0 and the paraboloid $z=x^2+y^2$ Then find the outward flux $\vec{F}$ across the boundary of the region D.

39. Let $\vec{F}=x^2\hat{i}-2xy\hat{j}+3xz\hat{k}$ and D: The region cut from the first octant by the sphere $x^2+y^2+z^2=4$, Then find the outward flux $\vec{F}$ across the boundary of the region D.

40. Let $\vec{F}=2xz\hat{i}-xy\hat{j}-z^2k$ and D: The wedge cut from the first octant by the plane $y+z=4$ and the elliptical cylinder $4x^2+y^2=16$. Then find the outward flux $\vec{F}$ across the boundary of the region D.

41. Let $\vec{F}=\sqrt{x^2+y^2+z^2}(x\hat{i}+y\hat{j}+z\hat{k})$ and D: The region $1\le x^2+y^2+z^2\le 2$. Then find the outward flux $\vec{F}$ across the boundary of the region D.

42. Let $\vec{F}=(5x^3+12xy^2)\hat{i}+(y^{3x}+e^y\sin z)\hat{j}+(5z^3+e^y\cos z)\hat{k}$ and D: The solid region between the sphere $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=2$. Then find the outward flux $\vec{F}$ across the boundary of the region D.

ANSWER KEY

1. -8π	**2.** $\frac{1}{3}$	**3.** 36
4. (0)	**5.** $4\pi a^3$	**6.** 12V
7.	**8.**	**9.** 18π
10. (a) 3, (b) $4\pi a^3$	**11.** 320π	**12.** (a) $\frac{8}{3}$, (b) $\frac{8}{3}(\hat{j}-\hat{k})$
13. 180	**14.** (a) 32π (b) 24 (c) 24π	**15.**
16. (–48)	**17.** $12(e-e^{-1}) = 6\sinh 1$	**18.** 3750π
19. (0)	**20.** $\left(\frac{1}{\pi}+\frac{5}{24}\right)$	**21.** $\left(\frac{\pi}{2}h^4\right)$
22. (14π)	**23.** (12)	**24.** (0)
25. 2π	26. $3\pi a^2$	**27.** 180π
28.	**29.**	**30.**
37. –16	**38.** $\mathbf{-8\pi}$	**39** $\mathbf{3\pi}$